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A generalization of the Lax equation

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Abstract

We propose a generalization of the standard Lax equation defined by means of an arbitrary action of a Lie algebra on a matrix differential manifold. We analyze properties of obtained equation and show examples with physical applications. In particular, certain constructions of Hamiltonian subclasses of this generalized Lax equation are described. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the early stages of classical mechanics it was the ultimate goal to integrate the equations of motion of as many physical systems as possible. One of the biggest achievements of those efforts is associated with works of Jacobi, who applied the method of separation of variables for Hamilton–Jacobi equations. During this epoch, it appeared possible to integrate explicitly or by quadratures all differential equations of motion.

Now, thanks to Poincaré we know that most Hamiltonian systems are not integrable and integrability is something exceptional. However, during the late sixties many new completely integrable systems were found. Among them there is a big class given by partial differential equations with infinitely many first integrals. Systems in this class have several characteristic properties: they are Hamiltonian ones in an infinite-dimensional function space, they have families of exact analytic solutions, the most striking of them are solitons. However, the most important property of these kind of equations is their connection with the spectral problem of linear differential operators. A relation between this class of

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non-linear partial differential equations and differential operators was explained by Lax [26]. He analyzed the KdV equation

$$u_t = 6uu_x - u_{xxx}, \quad x, t \in \mathbb{R}, \quad u(x,t) \in \mathbb{R}, \tag{1.1}$$

and observed that if we introduce two differential operators L and N:

$$L := \partial_x^2 - u(x,t), \qquad N := -4\partial_x^3 + 3(u\partial_x + \partial_x u), \quad \partial_t = \frac{\partial}{\partial t}, \quad \partial_x = \frac{\partial}{\partial x}, \tag{1.2}$$

then the operator equation

$$\dot{L} = [N, L] \tag{1.3}$$

is equivalent to KdV equation (1.1). Here $[\cdot,\cdot]$ denotes the commutator of differential operators. Then it is crucial to notice that the solution of (1.3) can be written in the form

$$L(t) = G(t)L_0G^{-1}(t), \tag{1.4}$$

where $L_0 = L(0)$ is the initial condition and $G(t)$ is the solution of the following initial-value problem

$$\dot{G}(t) = NG(t), \quad G(0) = \mathbb{1}. \tag{1.5}$$

This implies that if $u(t)$ is a solution of KdV equation, then spectrum of the operator $L(t)$ does not depend on t. A change of L in time is an isospectral deformation. As a result, the spectrum of $L(t)$ gives integrals of KdV equation. These considerations explained an observation of Gardner et al. [18] that eigenvalues of the Schrödinger operator with potential satisfying KdV equation are first integrals of this equation.

Till now many partial differential equations have been found, they can be written as Eq. (1.3) with appropriate differential operators N and L (see for instance [6,12]).

The idea of finding integrals of motion as eigenvalues of an associated operator L for systems with finitely many degrees of freedom

$$\dot{x}^i = F^i(t, x^1, \ldots, x^k), \quad i = 1, \ldots, k \tag{1.6}$$

in a phase space $\mathcal{P}$ of dimension k was developed by Flaschka [16,17] and Moser [29]. In this case operators L and N are just matrices whose entries depend on the dynamical variables x^i in such a way that differential equations for quantities $x^1, \ldots, x^k$ obtained from the matrix differential equation (1.3) have exactly the same form as in (1.6). Matrices L and M are called the Lax pair and Eq. (1.3) the Lax representation of (1.6).

The eigenvalues of L give first integrals of (1.3) and in consequence also (1.6). Instead of eigenvalues of L, it is more convenient to use other quantities which are functions of the spectrum of L, e.g. coefficients of the characteristic polynomial of L, or the quantities

$$I_j := \operatorname{Tr} L^j(t), \quad j \in \mathbb{N}. \tag{1.7}$$

The existence of the Lax representation of the analysed dynamical system is important for several reasons:

- It gives a set of first integrals of the analyzed system. However, their independence and involutivity (in the case of Hamiltonian systems) must be checked independently.
- In some cases the Lax representation has a form of Hamiltonian matrix differential equations on appropriate Lie algebra or Lie group. Then, obtained first integrals are pairwise in involution and this follows directly from the construction.

The application of Lax's idea allowed to simplify proofs of integrability (sometimes very complicated) of many systems and gave many new completely integrable systems both with finitely or infinitely many degrees of freedom. We can mention e.g. classical Calogero–Sutherland–Moser systems [8,29,31,33], the Toda lattice [42] and its different generalizations [16,17], the multidimensional Neumann system, equations of geodesic on an ellipsoid [30], different generalizations of n-dimensional rigid body [38].

Through this paper we restrict ourselves to finite-dimensional systems and, as a consequence, all analyzed operator objects have matrix character. This assumption guarantees the correctness of definition for all operations and simplifies proofs.

It is worth pointing that Lax equations appear also in quantum mechanics, e.g. the von Neumann equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H(t), \rho], \tag{1.8}$$

describing an evolution of quantum systems with finite-dimensional Hilbert space, has this form. Quantity ρ is the density matrix, and H the Hamiltonian of the system. From Lax's form of (1.8), it immediately follows that quantities

$$I_j := \mathrm{Tr}\, \rho^j(t), \quad j \in \mathbb{N}, \tag{1.9}$$

are first integrals for any quantum system described in the formalism of density matrices. This fact was not widely known among atomic physicists and even in the 1980s there appeared papers about n-level systems, in which authors notified a discovery of "a number of unexpected non-linear constants of motion", namely functions of the form (1.9).

The importance of the Lax equation in classical mechanics and the von Neumann equation in quantum mechanics stimulates searches for other matrix differential equations which have properties similar to the Lax equations. In this work we present a whole class of such equations.

These generalized equations can be useful both in classical and in quantum mechanics. In classical mechanics they can be used as representations of systems different from Lax representations. There is no general recipe for finding a map which transforms the analyzed system into Lax equation, in general we have to solve a system of algebraic equations determining entries of L and N. We also apply the same procedure during searches for another matrix representations. For certain classes of dynamical system it is easier to find a matrix representation which is different from Lax representation. This means that we can find another representation with N and L whose entries are simpler functions of dynamical variables than in any Lax representation.

On the other hand, in quantum mechanics recently there appear many generalizations of its standard formulation for description of quantum open systems with dissipation, for construction of non-linear quantum theory and so on. In these generalizations many new matrix differential equations appear. The determination of form of equations similar to Lax equation affords a possibility for a classification of matrix differential equations applied in quantum mechanics. This classification makes it possible to decide which properties of the analyzed quantum system are a simple consequence of the form of evolution equations of applied theory, and which are characteristic only for this system.

As mentioned above, the main aim of this work is to find the most general class of matrix equations which have properties similar to the Lax equation and to analyze them. At first, we have to be precise about what “properties similar to the Lax equation” mean and what geometrical structures are responsible for these properties? It appears that the fundamental role plays the fact that the construction of the Lax equation has a connection with Lie groups and Lie algebras theory. The existence of this connection generates all characteristic properties of the Lax equation. We can explain this connection in the following way. We denote by $\mathcal{M}$ a matrix differential manifold. If N belongs to an appropriate Lie algebra $\mathfrak{g}$, then a map $\rho : \mathfrak{g} \times \mathcal{M} \mapsto \mathrm{T}\mathcal{M}$, defined by

$$\rho(N, L) = NL - LN, \tag{1.10}$$

is an example of an action of $\mathfrak{g}$ on $\mathcal{M}$. In expression (1.10) we recognize the right-hand side of the Lax equation. We can write the solution of the Lax equation as a family of similarity transformations indexed by continuous parameter t:

$$\Phi(G(t), L_0) = G(t)L_0G^{-1}(t) \tag{1.11}$$

acting on an initial condition L_0. From the point of view of the Lie group theory, the map $\Phi(G, L_0) = GL_0G^{-1}$ is an example of action of $\mathcal{G}$, the Lie group of Lie algebra $\mathfrak{g}$, on $\mathcal{M}$. A possibility of writing a solution by means of an appropriate Lie group action implies that for any t, $L(t)$ belongs to some orbit of action Φ determined by L_0. This fact is of key importance in the theory of the Lax equation.

The form of the standard Lax equation is related to the so-called adjoint action. Following this we construct a generalization of the Lax equation by the matrix differential equation with the right-hand side defined by an arbitrary Lie algebra action. The solution of this generalized equation can be written in terms of an appropriate Lie group action. Then we present examples of such Lie algebra actions. In these examples the respective Lie group actions have the form of transformations acting on a fixed element from $\mathcal{M}$. We restrict our attention to the frequently used transformations of this type. Such a choice of the form of representations makes possible to use results obtained by people working in linear algebra over linear preserver problems [35]. Then we analyze the obtained equations: in particular we show sets of first integrals and constructions of Hamiltonian subclasses. We illustrate our considerations with a number of examples from classical and quantum mechanics.

The order of this paper is the following. In Section 2, we introduce basic notions and we present our generalization of the Lax equation, then we analyze general properties of this equation and its geometrical interpretation. In Section 3, we consider examples of our equation related to particular actions and we present some examples of physical equations of these types from classical and quantum mechanics. In Sections 4 and 5, we propose constructions of Hamiltonian matrix differential equations on a matrix manifold which has a Lie algebra and a Lie group structure, respectively.

2. Matrix differential equations built by means of an arbitrary Lie algebra action

We begin with setting up notation and terminology. Throughout this paper $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$ and $\mathcal{M}$ a matrix differential manifold. Let $\mathfrak{g}$ denote a Lie algebra. We assume that it is associated with a certain closed subgroup $\mathcal{G}$ of $\mathrm{GL}(n, \mathbb{K})$. By definition, an action ρ of $\mathfrak{g}$ on $\mathcal{M}$ is a smooth map

$$\rho : \mathfrak{g} \times \mathcal{M} \to \mathrm{T}\mathcal{M}, \qquad (N, L) \mapsto \rho(N, L),$$

such that the induced map $N \mapsto \rho_N$, where $\rho_N : \mathcal{M} \to \mathrm{T}\mathcal{M}$ defined by $\rho_N(L) := \rho(N, L)$, is a homomorphism of $\mathfrak{g}$ into the Lie algebra of vector fields $\mathcal{X}(\mathcal{M})$ on $\mathcal{M}$. It means that ρ_N fulfills the following conditions:

$$\forall\, N_1, N_2 \in \mathfrak{g}, \;\; \forall\, \alpha, \beta \in \mathbb{K}, \quad \rho_{\alpha N_1 + \beta N_2} = \alpha \rho_{N_1} + \beta \rho_{N_2}, \tag{2.1a}$$

$$\forall\, N_1, N_2 \in \mathfrak{g}, \quad \rho_{[N_1, N_2]} = |[\rho_{N_1}, \rho_{N_2}]|. \tag{2.1b}$$

The expression $|[\boldsymbol{X}, \boldsymbol{Y}]|$ denotes the standard Lie bracket in $\mathcal{X}(\mathcal{M})$ which for any $\boldsymbol{X}(L)$, $\boldsymbol{Y}(L) \in \mathrm{T}_L\mathcal{M}$ is defined by

$$|[\boldsymbol{X}, \boldsymbol{Y}]|(L) := \mathrm{T}\,\boldsymbol{X}(L)(\boldsymbol{Y}(L)) - \mathrm{T}\,\boldsymbol{Y}(L)(\boldsymbol{X}(L)). \tag{2.2}$$

Following Abraham and Marsden [1] we denote by $\mathrm{T}\,\boldsymbol{X}$ the differential of the map $\boldsymbol{X}$, which any L assigns $\boldsymbol{X}(L)$. In our examples we meet the following actions of Lie algebra $\mathfrak{g}$:

$$\begin{aligned} &\mathrm{ad}(N, L) = NL - LN, \qquad \mathrm{cq}(N, L) = \bar{N}L - LN, \qquad \mathrm{sq}(N, L) = NL + LN^{\mathrm{T}}, \\ &\mathrm{hq}(N, L) = NL + LN^{\dagger}, \qquad \mathrm{dq}(N_1, N_2; L) = N_1 L - L N_2. \end{aligned} \tag{2.3}$$

In the last example we have in fact the action of the product $\mathfrak{g} \times \mathfrak{g}$ on $\mathcal{M}$. The expression $\bar{N}$, N^T, and $N^{\dagger}$ denotes the complex conjugation, transposition and Hermitian conjugation of matrix N, respectively.

Now we can write our generalization of the Lax equation

$$\dot{L} = \rho_{N(t,L)}(L), \quad L(0) = L_0, \tag{2.4}$$

where dependence $N = N(t, L)$ means that we admit arbitrary dependence of entries of N on entries of L and t. We only assume that $N \in \mathfrak{g}$ for all t.

In order to understand the geometrical interpretation of (2.4), we introduce several notions from representation theory of Lie groups and Lie algebras. By definition, an action Φ of a Lie group $\mathcal{G}$ on $\mathcal{M}$ is a smooth mapping

$$\Phi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}, \qquad (G, L) \mapsto \Phi(G, L),$$

which satisfies the following conditions:

$$\forall G_1, G_2 \in \mathcal{G},\ \forall L \in \mathcal{M},\quad \Phi(G_2G_1, L) = \Phi(G_2, \Phi(G_1, L)), \tag{2.5a}$$

$$\forall G \in \mathcal{G},\ \forall L \in \mathcal{M},\quad \Phi(\mathbb{1}, L) = L, \tag{2.5b}$$

where $\mathbb{1}$ denotes the identity element in $\mathcal{G}$. Using Φ we can build two new maps $\Phi_L : \mathcal{G} \to \mathcal{M}$ and $\Phi_G : \mathcal{M} \to \mathcal{M}$ defined by $G \mapsto \Phi(G, L)$ and $L \mapsto \Phi(G, L)$, respectively. The two conditions (2.5a) and (2.5b) can now be reformulated by saying that the map $G \mapsto \Phi_G$ is a homomorphism of the Lie group $\mathcal{G}$ into $\mathrm{Diff}(\mathcal{M})$, the group of diffeomorphisms of $\mathcal{M}$.

It is well known that with each Lie group $\mathcal{G}$ is associated the Lie algebra $\mathfrak{g}$. For our needs the relation between $\mathcal{G}$ and $\mathfrak{g}$ can be expressed in the following theorem.

Theorem 1 (Watkins and Elsner [43]). *If $G(t)$ is the solution of the initial-value problem*

$$\dot{G}(t) = NG(t),\quad G(0) = \mathbb{1}, \tag{2.6}$$

then $G(t) \in \mathcal{G}$ for all t if and only if $N \in \mathfrak{g}$.

In the above theorem, it is possible to use another initial-value problem

$$\dot{G}'(t) = G'(t)N,\quad G(0) = \mathbb{1}, \tag{2.7}$$

because its solution coincides with the solution of (2.6). Watkins and Elsner [43] generalized the theorem about the connection between Lie groups and Lie algebras to the case when N depends on t.

Theorem 2 (Watkins and Elsner, Theorem 5.1^T in [43]). *Let $N(t)$ be a continuous function from $[0, t_1]$ into $\mathbb{K}^{n\times n}$ and let $G(t)$ be the solution of an initial-value problem*

$$\dot{G} = N(t)G,\quad G(t_0) \in \mathcal{G} \tag{2.8}$$

on $[t_0, t_1]$, where $\mathcal{G}$ is any closed subgroup of $GL(n, \mathbb{K})$. Then $G(t) \in \mathcal{G}$ for all $t \in [t_0, t_1]$ if and only if $N(t) \in \mathfrak{g}$ for all $t \in [t_0, t_1]$.

There exists a companion to Theorem 2 about relations between $\mathcal{G}$ and $\mathfrak{g}$ based on the next initial-value problem

$$\dot{G}' = G'N(t),\qquad G'(t_0) \in \mathcal{G}. \tag{2.9}$$

Both Lie groups and Lie algebras act on $\mathcal{M}$. Furthermore, we know that every Lie group has its Lie algebra and that every action of a Lie group induces appropriate action of related Lie algebra. If we think of action Φ on $\mathcal{M}$ as a homomorphism of $\mathcal{G}$ into the group of diffeomorphisms of $\mathcal{M}$, defined by $G \mapsto \Phi_G$, we can calculate the differential of this map at the identity element $\mathbb{1} \in \mathcal{G}$. In this way we obtain the induced homomorphism of Lie algebra $\mathfrak{g}$ into the Lie algebra of vector fields on $\mathcal{M}$. It is exactly the map $N \mapsto \rho_N$, where ρ_N is calculated as

$$\rho_N(L) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_{\exp\{sN\}}(L) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_L(\mathrm{e}^{sN}) = \mathrm{T}_{\mathbb{1}}\Phi_L(N). \tag{2.10}$$

Vector ρ_N is sometimes called an infinitesimal generator of action Φ.

We see that having a form of Φ we can always calculate ρ but if we have ρ we cannot in general find its associated action Φ. This last operation is called an integration of a Lie algebra action. We restrict ourselves to cases when it is possible to integrate actions of a Lie algebra to obtain its associated Lie group actions. An integration of Lie algebra actions from examples (2.3) gives the following results:

$$\mathrm{Ad}(G,L)=GLG^{-1},\qquad \mathrm{Cq}(G,L)=\bar{G}LG^{-1},\qquad \mathrm{Sq}(G,L)=GLG^{\mathrm{T}},$$
$$\mathrm{Hq}(G,L)=GLG^{\dagger},\qquad \mathrm{Dq}(G_1,G_2;L)=G_1LG_2^{-1}, \tag{2.11}$$

respectively.

The next key object for our considerations are immersed subsets of $\mathcal{M}$ — the so-called orbits. If Φ is an action of $\mathcal{G}$ on $\mathcal{M}$, the orbit of this action passing through $L_0\in\mathcal{M}$ is defined by

$$\mathcal{O}_{L_0}:=\{\Phi_G(L_0),\,G\in\mathcal{G}\}. \tag{2.12}$$

The tangent space at $L\in\mathcal{O}_{L_0}$ to orbit $\mathcal{O}_{L_0}$ is given by

$$\mathrm{T}_L\mathcal{O}_{L_0}=\{\rho_N(L)|N\in\mathfrak{g}\}. \tag{2.13}$$

As $L\in\mathcal{O}_{L_0}$, we can write

$$L=\Phi_G(L_0) \tag{2.14}$$

for some $G\in\mathcal{G}$. We note that the tangent vectors to the orbit $\mathcal{O}_{L_0}$ at L have the same forms as the right-hand side of our matrix differential equation (2.4). As a consequence, orbit $\mathcal{O}_{L_0}$ is invariant with respect to the phase flow generated by (2.4) and the solution of the initial-value problem

$$\dot{L}=\rho_{N(t)}(L),\quad L(0)=L_0, \tag{2.15}$$

where $N(t)=N(t,L(t))$ is a curve in $\mathfrak{g}$, can be written as

$$L(t)=\Phi_{G(t)}(L_0). \tag{2.16}$$

The map Φ is the action of Lie group $\mathcal{G}$ associated with action ρ of Lie algebra $\mathfrak{g}$. The function $G(t)$ determines a curve in $\mathcal{G}$. This curve is defined by function $N(t)$, the curve in $\mathfrak{g}$. Now we have to find a relation between these two curves. We know that one curve $N(t)$ in Lie algebra $\mathfrak{g}$ generates two curves $G(t)$, $G'(t)$ in the related Lie group $\mathcal{G}$

$$\dot{G}(t)=N(t)G(t),\quad G(0)=\mathbb{1}, \tag{2.17a}$$

$$\dot{G}'(t)=G'(t)N(t),\quad G'(0)=\mathbb{1}. \tag{2.17b}$$

From the introduction we know that for the Lax equation

$$\dot{L}=[N(t),L], \tag{2.18}$$

this relation has the form (2.17a). Only such relation guarantees that if we differentiate equation

$$L(t) = G(t)L_0G^{-1}(t) \tag{2.19}$$

with respect to t and use (2.17a), then we obtain (2.18). If we apply the second relation (2.17b), we also obtain the Lax equation but with transformed matrix $\tilde{N} = G(t)N(t, L)G^{-1}(t)$. The relation (2.17a) is a condition of agreement for (2.18) and (2.19).

In order to find such condition of agreement for Eqs. (2.15) and (2.16), we differentiate (2.16) with respect to t. We can rewrite the expression (2.16) determining $L(t)$ using the map Φ_{L_0}:

$$L(t) = \Phi_{G(t)}(L_0) = \Phi_{L_0}(G(t)). \tag{2.20}$$

Taking derivative with respect to the parameter t, we find that

$$\dot{L}(t) = \mathrm{T}_{G(t)}\Phi_{L_0}(\dot{G}(t)), \tag{2.21}$$

where $\dot{G}(t) \in \mathrm{T}_{G(t)}\mathcal{G}$, $\dot{L}(t) \in \mathrm{T}_{L(t)}\mathcal{M}$.

In order to introduce an action ρ of the associated Lie algebra $\mathfrak{g}$ we replace the derivative $\mathrm{T}_{G(t)}\Phi_{L_0}$ at point $G(t)$ by the derivative of the same map at $\mathbb{1} \in \mathcal{G}$. If we use the right translation R_G on $\mathcal{G}$, and we recall (2.20), then we find

$$\begin{aligned}\Phi_{L(t)}(G') &= \Phi(G', L(t)) = \Phi(G', \Phi_G(L_0)) \\ &= \Phi(G', \Phi(G, L_0)) = \Phi(G'G, L_0) = (\Phi_{L_0} \circ \mathrm{R}_G)(G'),\end{aligned} \tag{2.22}$$

or in the operator form

$$\Phi_{L(t)} = \Phi_{L_0} \circ \mathrm{R}_G, \tag{2.23}$$

where $\circ$ denotes the superposition of maps. We can compute the derivative of $\Phi_{L(t)}$ using the chain rule:

$$\mathrm{T}\Phi_{L(t)} = \mathrm{T}\Phi_{L_0} \circ \mathrm{T}\mathrm{R}_G. \tag{2.24}$$

From the above relation we obtain the expression

$$\mathrm{T}\Phi_{L_0} = \mathrm{T}\Phi_{L(t)} \circ \mathrm{T}\mathrm{R}_{G^{-1}}, \tag{2.25}$$

which we put into (2.21). We find the following expression on $\dot{L}(t)$

$$\dot{L}(t) = \mathrm{T}_{\mathbb{1}}\Phi_{L(t)}(\mathrm{T}_{G(t)}\mathrm{R}_{G^{-1}(t)}(\dot{G}(t))). \tag{2.26}$$

Since $\mathrm{R}_{G^{-1}(t)}$ is the right translation leading $G(t)$ to the identity element $\mathbb{1} \in \mathcal{G}$ and $\dot{G}(t) \in \mathrm{T}_{G(t)}\mathcal{G}$, it follows that $\mathrm{T}_{G(t)}\mathrm{R}_{G^{-1}(t)}(\dot{G}(t)) \in \mathrm{T}_{\mathbb{1}}\mathcal{G} = \mathfrak{g}$. Moreover, for $\mathcal{G}$ as a subgroup of $GL(n, \mathbb{K})$, the right translation reduces to right multiplication by the corresponding matrix, and hence

$$\mathrm{T}_{G(t)}\mathrm{R}_{G^{-1}(t)}(\dot{G}(t)) = \dot{G}(t)G^{-1}(t) \in \mathfrak{g}.$$

It means that there exists such $N(t) \in \mathfrak{g}$ that

$$\dot{G}(t)G^{-1}(t) = N(t). \tag{2.27}$$

In this equation we recognize (2.17a). Now, we can rewrite (2.26) in the following form:

$$\dot{L}(t) = \mathrm{T}_{\mathbb{1}}\Phi_{L(t)}(N(t)) = \rho_{N(t)}(L). \tag{2.28}$$

The last equality follows from the definition of ρ_N. We see that the solution of (2.15) has the form (2.16) if $N(t)$ defines $G(t)$ in the way (2.17a). We can recapitulate above considerations in the following theorem.

Theorem 3. *Let* $\mathfrak{g}$ *be a Lie algebra of a certain closed Lie subgroup* $\mathcal{G}$ *of* $\mathrm{GL}(n, \mathbb{K})$ *and* ρ *an arbitrary action of* $\mathfrak{g}$ *on* $\mathcal{M}$. *Then the initial-value problem*

$$\dot{L} = \rho_{N(t,L)}(L), \quad L(0) = L_0, \tag{2.29}$$

where $L, L_0 \in \mathcal{M}$, *and* $N = N(t, L) \in \mathfrak{g}$ *has a unique solution which we can write as*

$$L(t) = \Phi_{G(t)}(L_0), \tag{2.30}$$

where the function $G(t)$ *is the solution of*

$$\dot{G}(t) = N(t, L)G(t), \quad G(0) = \mathbb{1}. \tag{2.31}$$

The map Φ *is the action of* $\mathcal{G}$ *corresponding to action* ρ *of* $\mathfrak{g}$.

Proof. To see that L satisfies (2.30), assume that $\hat{L} = \Phi_{G(t)}(L_0)$. To show that $\hat{L}$ fulfills (2.29) we differentiate $\hat{L}$ following calculations (2.21)–(2.28). By the uniqueness of the solution $\hat{L} = L$. □

Now we introduce a notion of a $\mathcal{G}$-invariant function. Function $\varphi \in C^{\infty}(\mathcal{M})$ is a $\mathcal{G}$-invariant function of an action Φ of Lie group $\mathcal{G}$ if it fulfills the condition

$$\forall\, G \in \mathcal{G}, \ \ \forall\, L \in \mathcal{M}, \quad \varphi(\Phi_G(L)) = \varphi(L). \tag{2.32}$$

It means that a $\mathcal{G}$-invariant function is constant on the orbits of action Φ.

The $\mathcal{G}$-invariant functions of Φ have a connection with the dynamic described by (2.29).

Theorem 4. *The* $\mathcal{G}$*-invariant functions of* Φ *are first integrals of* (2.29).

Proof. The proof of this theorem is based on the observation that the evolution described by (2.29) is confined to a certain orbit determined by L_0. □

3. Examples of the generalized Lax equation

In this section we present five examples of our generalized Lax equation. We divide the first four examples into two classes related to automorphisms and anti-automorphisms of $\mathbb{K}^{n\times n}$. This procedure is useful because of the following:

1. It enlarges the class of matrix differential equations which we can include in our analysis. In fact, we can use any automorphism of an arbitrary order and any anti-automorphism of an arbitrary even order. We present in detail only the most popular automorphisms and anti-automorphisms.
2. There exists one construction of Hamiltonian equations on a Lie group for any automorphism (anti-automorphism).

An automorphism τ of $\mathbb{K}^{n\times n}$ is a map $\tau : \mathbb{K}^{n\times n} \to \mathbb{K}^{n\times n}$ which fulfills the following conditions:

$$\forall\, X_1, X_2 \in \mathbb{K}^{n\times n}, \quad \tau(X_1 + X_2) = \tau(X_1) + \tau(X_2), \tag{3.1a}$$

$$\forall\, X_1, X_2 \in \mathbb{K}^{n\times n}, \quad \tau(X_1 X_2) = \tau(X_1)\tau(X_2). \tag{3.1b}$$

From (3.1b), we can deduce that for non-singular X

$$\tau(X^{-1}) = \tau^{-1}(X). \tag{3.2}$$

Using any automorphism of $\mathbb{K}^{n\times n}$, we can always build the action of a certain Lie algebra $\mathfrak{g}$

$$\rho^{\tau}(N, L) = \tau(N)L - LN, \quad N \in \mathfrak{g}, \tag{3.3}$$

and associated matrix differential equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \tau(N(t, L))L - LN(t, L), \quad L(0) = L_0. \tag{3.4}$$

An integration of action ρ^{τ} gives

$$\Phi^{\tau}(G, L) = \tau(G)LG^{-1}, \quad G \in \mathcal{G}, \tag{3.5}$$

and we can write the solution of (3.4) as

$$L(t) = \tau(G(t))L_0 G^{-1}(t), \tag{3.6}$$

where matrix G satisfies the following matrix differential equation on $\mathcal{G}$:

$$\dot{G}(t) = N(t, L)G(t), \qquad G(0) = \mathbb{1}. \tag{3.7}$$

If we assume that an automorphism has order q, i.e.

$$\forall X \in \mathbb{K}^{n\times n}, \quad \tau^{q}(X) = X \tag{3.8}$$

for a certain $q \in \mathbb{N}$, then it is easy to obtain the set of $\mathcal{G}$-invariant functions of Φ^{τ} and Theorem 4 yields the following results.

Theorem 5. *Eigenvalues of*

$$\tau^{q-1}(L(t))\tau^{q-2}(L(t))\cdots\tau^2(L(t))\tau(L(t))L(t)$$

are first integrals of (3.4).

Proof. We calculate $\tau^s(L(t))$ using (3.6) and the properties of an automorphism of order q

$$\tau^s(L(t)) = \tau^{s+1}(G(t))\tau^s(L_0)\tau^s(G^{-1}(t)), \quad s = 0, 1, \dots, q-1. \tag{3.9}$$

From (3.6) and (3.9), it is obvious that for any $t \geq 0$, the quantity

$$\tau^{q-1}(L(t))\tau^{q-2}(L(t))\cdots\tau^2(L(t))\tau(L(t))L(t)$$

is connected with

$$\tau^{q-1}(L_0)\tau^{q-2}(L_0)\cdots\tau^2(L_0)\tau(L_0)L_0$$

by means of a similarity transformation. □

Matrix differential equations of the form (3.4) and a formulation of Theorem 5 can be found in [4,5], where the author applied another approach without Lie group and Lie algebra interpretation.

The most popular are two automorphisms of the second order: the identity transformation $\tau(X) = X$ and the complex conjugation $\tau(X) = \bar{X}$. In the first case, Eq. (3.4) transforms into the well-known Lax equation. In accordance with the above considerations, eigenvalues of L^2 are first integrals of this equation. In fact, eigenvalues of any power of L are constants during the evolution described by the Lax equation.

Now we present a few physical examples of the Lax equation.

Example 1. Two generalizations of the von Neumann equation:

1. a family of equations introduced by Gisin [19,20]:

$$\dot{\rho} = -\frac{\mathrm{i}}{\hbar}[H, \rho] - \frac{k}{\hbar^2}[[H, \rho], \rho] = \left[-\frac{\mathrm{i}}{\hbar}H - \frac{k}{\hbar^2}[H, \rho], \rho\right], \tag{3.10}$$

   where k is an arbitrary positive parameter,
2. a family of equations introduced by Czachor [10,11]:

$$\dot{\rho} = -\frac{\mathrm{i}}{\hbar}[H, \rho^k], \quad k \in \mathbb{N}. \tag{3.11}$$

The Lax's form of Eq. (3.11) is apparent if we introduce new matrix $\tilde{N}$:

$$\tilde{N} := \sum_{j=0}^{k-1}\rho^j H \rho^{k-1-j}, \tag{3.12}$$

because

$$[H, \rho^k] = [\tilde{N}(\rho), \rho].$$

Example 2. Each completely integrable (in the Liouville's sense) Hamiltonian system admits the Lax representation in a neighborhood of its invariant tori [2,24]. This representation is expressed in the action–angle variables $(I_1, \dots, I_m, \varphi_1, \dots, \varphi_m)$. We can choose the following box-diagonal matrices L and N:

$$L = \begin{pmatrix} I_1 & & & & & \\ & \ddots & & & & \\ & & I_m & & & \\ & & & L_1 & & \\ & & & & \ddots & \\ & & & & & L_m \end{pmatrix}, \quad N = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & N_1 & & \\ & & & & \ddots & \\ & & & & & N_m \end{pmatrix}, \tag{3.13}$$

having boxes of the forms

$$L_j = \begin{pmatrix} 0 & e^{i\varphi_j} \\ 0 & 0 \end{pmatrix}, \qquad N_j = \begin{pmatrix} i\omega_j & 0 \\ 0 & 0 \end{pmatrix}, \quad j = 1, \dots, m. \tag{3.14}$$

We constructed this representation following Kozlov [24]. The eigenvalues of L are $I_1, \dots, I_m$ and 0 with multiplicity $2m$.

Example 3. Let T be tensor invariant of the valence (1,1) for an autonomous dynamical system

$$\dot{x}^i = F^i(x^1, \dots, x^k), \quad i = 1, \dots, k. \tag{3.15}$$

Then equation which defines a tensor invariant

$$\mathcal{L}_F T = 0, \quad \mathcal{L}: \text{ the Lie derivative}, \tag{3.16}$$

has the form of the Lax equation with the following matrices L and N:

$$L^{ij} := T^i_j, \qquad N^i_j := \frac{\partial F^i}{\partial x^j}. \tag{3.17}$$

The symbols L^{ij} and N^i_j denote the elements in the ith rows and jth columns of the matrices L and N, respectively. This property for bi-Hamiltonian systems [3] with two different symplectic forms (from which we can construct the tensor invariant of the valence (1,1)) was observed at the beginning of the 1980s by e.g. de Filippo et al. [14,15], and Cariñena and Ibort [9].

Furthermore, there exist Lax representations for many autonomous dynamical systems. For review see [34].

The next example of the automorphism is the complex conjugation $\tau(X) = \bar{X}$. In this case Eq. (3.4) transforms into

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \bar{N}(t, L)L - LN(t, L). \tag{3.18}$$

We can write its solution with the initial condition $L(0) = L_0$ as

$$L(t) = \bar{G}(t)L_0 G^{-1}(t), \tag{3.19}$$

where $G(t)$ satisfies (3.7). From Theorem 5, we conclude that eigenvalues of $\bar{L}(t)L(t)$ are first integrals of (3.18).

Example 4. For every even integer $r = 2s$, the dynamical systems

$$\dot{a}_i = -a_i \left(m_{\alpha^{2s}(i)} \prod_{k=1}^{2s-1} \tau^k(a_{\alpha^k(i)}) - m_i \prod_{k=1}^{2s-1} \tau^k(a_{\alpha^{-k}(i)}) \right), \quad r = 2s > 0, \tag{3.20a}$$

$$\dot{\underline{a}}_i = -\underline{a}_i \left(m_i \prod_{k=0}^{2s} \tau^k(\underline{a}_{\alpha^k(i)}) - m_{\alpha^{-2s}(i)} \prod_{k=0}^{2s} \tau^k(\underline{a}_{\alpha^{-k}(i)}) \right), \quad r = -2s < 0, \tag{3.20b}$$

have the matrix representations (3.18). Here $\tau(X) = \bar{X}$, $i = 1, \ldots, n$, α is an arbitrary permutation of $\{1, 2, \ldots, n\}$ and m_i are arbitrary constants. For these two systems, non-zero entries of matrices L and N are the following:

$$L_{i\alpha(i)} = a_i, \qquad L_{i\gamma(i)} = m_i, \qquad N_{i\beta(i)} = -\bar{x}_i. \tag{3.21}$$

The permutations β and γ are defined as

$$\beta = \alpha^r, \qquad \gamma = \alpha\beta^{-1}. \tag{3.22}$$

Quantities x_i in both cases are equal

$$x_i = \prod_{k=1}^{2s-1} \tau^k(a_{\alpha^k(i)}), \qquad \tau(a) = \bar{a}, \quad r = 2s > 0, \tag{3.23a}$$

$$x_i = \prod_{k=0}^{2s} \tau^k(\underline{a}_{\alpha^{-k}(i)}), \qquad \underline{a}_i = a_i^{-1}, \quad r = -2s < 0, \tag{3.23b}$$

respectively. This example is due to Bogoyavlensky [4].

The next class of examples is associated with anti-automorphisms κ of $\mathbb{K}^{n\times n}$. Let us recall that an anti-automorphism κ of $\mathbb{K}^{n\times n}$ is a map $\kappa : \mathbb{K}^{n\times n} \to \mathbb{K}^{n\times n}$ which satisfies the following conditions:

$$\forall X_1, X_2 \in \mathbb{K}^{n\times n}, \quad \kappa(X_1 + X_2) = \kappa(X_1) + \kappa(X_2), \tag{3.24a}$$

$$\forall X_1, X_2 \in \mathbb{K}^{n\times n}, \quad \kappa(X_1 X_2) = \kappa(X_2)\kappa(X_1). \tag{3.24b}$$

By means of an arbitrary anti-automorphism, we can construct the action of a certain Lie algebra $\mathfrak{g}$:

$$\rho^{\kappa}(N, L) = NL + L\kappa(N), \quad N \in \mathfrak{g}. \tag{3.25}$$

An integration of this Lie algebra action gives appropriate Lie group action

$$\Phi^{\kappa}(G, L) = GL\kappa(G), \quad G \in \mathcal{G}. \tag{3.26}$$

In accordance with our general considerations, we can conclude that a solution of the equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = N(t, L)L + L\kappa(N(t, L)), \quad L(0) = L_0, \tag{3.27}$$

can be written as

$$L(t) = G(t)L_0\kappa(G(t)). \tag{3.28}$$

Here $G(t)$ is the solution of the initial-value problem

$$\dot{G}(t) = N(t, L)G(t), \quad G(0) = \mathbb{1}. \tag{3.29}$$

We assume that L_0 is non-singular. This assumption guarantees that for any $t \geq 0$, matrix $L(t)$ is non-singular. If we assume that κ has even order q, i.e.

$$\forall X \in \mathbb{K}^{n\times n}, \quad \kappa^q(X) = X \tag{3.30}$$

for a certain even $q \in \mathbb{N}$, then we can identify the set of $\mathcal{G}$-invariant functions of Φ^{κ}. Using the known relation between $\mathcal{G}$-invariant functions of Φ and the dynamics of the matrix differential equation defined by ρ, we can conclude the following theorem.

Theorem 6. *Eigenvalues of*

$$L(t)\kappa(L^{-1}(t))\kappa^2(L(t))\kappa^3(L^{-1}(t))\cdots\kappa^{q-2}(L(t))\kappa^{q-1}(L^{-1}(t))$$

are first integrals of (3.27).

Proof. Using (3.28) and properties of an anti-automorphism we obtain

$$\begin{aligned}
&\kappa^s(L^{-1}(t)) = \kappa^{s+1}(G^{-1}(t))\kappa^s(L_0^{-1})\kappa^s(G^{-1}(t)) \quad \text{for even } s \in \{0, 2, \dots, q\},\\
&\kappa^s(L^{-1}(t)) = \kappa^s(G^{-1}(t))\kappa^s(L_0^{-1})\kappa^{s+1}(G^{-1}(t)) \quad \text{for odd } s \in \{1, 3, \dots, q-1\},\\
&\kappa^s(L(t)) = \kappa^s(G(t))\kappa^s(L_0)\kappa^{s+1}(G(t)) \quad \text{for even } s \in \{0, 2, \dots, q\},\\
&\kappa^s(L(t)) = \kappa^{s+1}(G(t))\kappa^s(L_0)\kappa^s(G(t)) \quad \text{for odd } s \in \{1, 3, \dots, q-1\}.
\end{aligned} \tag{3.31}$$

If we define

$$M(t) = L(t)\kappa(L^{-1}(t))\kappa^2(L(t))\kappa^3(L^{-1}(t))\cdots\kappa^{q-2}(L(t))\kappa^{q-1}(L^{-1}(t)), \tag{3.32}$$

then from the above formulae it follows that

$$M(t) = G(t)M(0)G^{-1}(t). \tag{3.33}$$

As a consequence eigenvalues of $M(t)$ are constants during evolution (3.27). □

The properties of Eq. (3.27) for anti-automorphisms of the second order were analyzed in [36].

An example of the second order anti-automorphism of $\mathbb{K}^{n\times n}$ is the transposition of matrices $\kappa(X) = X^{\mathrm{T}}$. In this case, Eq. (3.27) transforms into

$$\frac{\mathrm{d}L}{\mathrm{d}t} = N(t, L)L + LN^{\mathrm{T}}(t, L), \qquad L(0) = L_0. \tag{3.34}$$

We can write its solution as

$$L(t) = G(t)L_0G^{\mathrm{T}}(t), \tag{3.35}$$

where $G(t)$ is defined by $N(t)$ in the standard way (3.29). From Theorem 6, we conclude that the eigenvalues of $L(L^{-1})^{\mathrm{T}}$ are first integrals of (3.34). It is worth mentioning that if L is a symmetric or skew-symmetric matrix, then all eigenvalues of $L(L^{-1})^{\mathrm{T}}$ are equal to 1 or -1, respectively, and in these cases we obtain only trivial first integrals. Now we present one physical example of the matrix differential equation with transposition.

Example 5. Let T be a tensor invariant of valence (2,0) for an autonomous system

$$\dot{x}^i = F^i(\boldsymbol{x}), \quad i = 1, \dots, k, \tag{3.36}$$

then the equation defining tensor invariant

$$\mathcal{L}_F T = 0 \tag{3.37}$$

has the form (3.34) with the following matrices L and N:

$$L_{ij} = T_{ij}, \qquad N^i_j := \frac{\partial F^i}{\partial x^j}. \tag{3.38}$$

Here the symbol N^i_j denotes the element in the ith row and jth column of N. An analogous statement is valid for tensor invariant of valence (0,2) with N given by

$$N^i_j := -\frac{\partial F^j}{\partial x^i}. \tag{3.39}$$

A relation between tensor invariants of valence (2,0) (or (0,2)) and the matrix differential equation with transposition was observed in [36].

The next example of the second order anti-automorphism is the Hermitian conjugation $\kappa(X) = X^{\dagger}$. Now Eq. (3.27) has the form

$$\frac{\mathrm{d}L}{\mathrm{d}t} = N(t, L)L + LN^{\dagger}(t, L), \qquad L(0) = L_0, \tag{3.40}$$

and we can write its solution as

$$L(t) = G(t)L_0G^{\dagger}(t). \tag{3.41}$$

Matrix $G(t)$ is determined by $N(t)$ in the standard way (3.29).

Example 6. Lamb [25,44] introduced the following generalization of the von Neumann equation:

$$\dot{\rho}(t) = -\frac{\mathrm{i}}{\hbar}[H, \rho(t)] - \frac{1}{2}\{\rho(t), \Gamma\}, \tag{3.42}$$

for description of certain excited states of hydrogen atom. Later this equation was applied in the theory of masers and lasers. In (3.42) $\{\cdot, \cdot\}$ denotes anti-commutator of matrices, H is a Hamiltonian of the system and Γ is a Hermitian matrix modeling dissipation. If we introduce a new non-Hermitian matrix $\tilde{H}$

$$\tilde{H} = -\frac{\mathrm{i}}{\hbar}H - \frac{1}{2}\Gamma, \tag{3.43}$$

then (3.42) can be written in the form (3.40). We present this example in order to draw attention to the fact that sometimes all first integrals calculated from Theorem 6 are trivial. In this example it follows from the fact that the density matrix is always Hermitian and $\rho(\rho^{-1})^{\dagger} = \mathbb{1}$.

By means of an arbitrary automorphism τ and an arbitrary anti-automorphism κ of $\mathbb{K}^{n\times n}$ we can build two actions of $\mathfrak{g}$ on $\mathcal{M}$. The first has the form

$$\rho^{\tau\kappa}(N, L) = \tau(N)L + L\kappa(N). \tag{3.44}$$

After integration we have appropriate action of $\mathcal{G}$ on $\mathcal{M}$

$$\Phi^{\tau\kappa}(G, L) = \tau(G)L\kappa(G). \tag{3.45}$$

Since it is difficult to find $\mathcal{G}$-invariant functions of action $\Phi^{\tau\kappa}$, we restrict ourselves to cases when τ and κ have order two. Then the following theorem describes properties of matrix differential equation (3.44).

Theorem 7. *Let τ and κ be an arbitrary second order automorphism and an anti-automorphism of $\mathbb{K}^{n\times n}$, respectively, and commute with each other. Then the solution of*

$$\dot{L} = \tau(N(t, L))L + L\kappa(N(t, L)), \quad L(0) = L_0, \tag{3.46}$$

can be written as

$$L(t) = \tau(G(t))L_0\kappa(G(t)), \tag{3.47}$$

where $G(t)$ is the solution of the following initial-value problem:

$$\dot{G}(t) = N(t, L)G(t), \quad G(0) = \mathbb{1}. \tag{3.48}$$

Furthermore, eigenvalues of $\kappa(L)\tau(L^{-1})$ are first integrals of (3.46).

Proof. The first part of the theorem follows from the forms of actions (3.44) and (3.45). From (3.47) and properties of τ and κ we obtain

$$\kappa(L) = G\kappa(L_0)\kappa(\tau(G)), \qquad L^{-1} = \kappa^{-1}(G)L_0^{-1}\tau^{-1}(G),$$
$$\tau(L^{-1}) = \tau(\kappa^{-1}(G))\tau(L_0^{-1})G^{-1}. \tag{3.49}$$

Using the above formulae, we calculate

$$\begin{aligned}\kappa(L(t))\tau(L^{-1}(t)) &= G(t)\kappa(L_0)\kappa(\tau(G(t)))\tau(\kappa^{-1}(G(t)))\tau(L_0^{-1})G^{-1}(t)\\ &= G(t)\kappa(L_0)\kappa(\tau(G(t)))\kappa^{-1}(\tau(G(t)))\tau(L_0^{-1})G^{-1}(t)\\ &= G(t)\kappa(L_0)\tau(L_0^{-1})G^{-1}(t),\end{aligned} \tag{3.50}$$

and now it is obvious that eigenvalues of $\kappa(L(t))\tau(L^{-1}(t))$ are constants. □

The second type of action ρ with any τ and κ is the following:

$$\rho^{\tau\kappa}(N, L) = NL + L\kappa(\tau(N)). \tag{3.51}$$

An integration of this action gives

$$\Phi^{\tau\kappa}(G, L) = GL\kappa(\tau(G)). \tag{3.52}$$

From general considerations about matrix differential equations defined by Lie group and Lie algebra actions, we can deduce the next theorem.

Theorem 8. *Let τ and κ be any second order automorphism and anti-automorphism of $\mathbb{K}^{n\times n}$, respectively, and commute with each other. Then the solution of*

$$\dot{L} = N(t, L)L + L\kappa(\tau(N(t, L))) \tag{3.53}$$

with the initial condition $L(0) = L_0$ has the following form:

$$L(t) = G(t)L_0\kappa(\tau(G(t))). \tag{3.54}$$

Here $G(t)$ is the solution of the initial-value problem on $\mathcal{G}$:

$$\dot{G}(t) = N(t, L)G(t), \quad G(0) = \mathbb{1}. \tag{3.55}$$

Furthermore, eigenvalues of $L\kappa(\tau(L^{-1}))$ are first integrals of (3.46).

Proof. The proof is analogous to that of the previous theorem. □

It is worth noticing that in constructions of all analyzed equations defined by means of τ (of form (3.4)), κ (of form (3.27)) and a pair τ and κ (of form (3.46) and (3.53)), we can sometimes use automorphisms and anti-automorphisms of certain subsets of $\mathbb{K}^{n\times n}$. Namely, if $\mathcal{M}$ has a structure of Lie group $\mathcal{G}$ of acting Lie algebra $\mathfrak{g}$, then we can use automorphisms and anti-automorphisms of $\mathfrak{g}$. A map $\tau : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of Lie algebra $\mathfrak{g}$ if it satisfies relations

$$\forall N_1, N_2 \in \mathfrak{g}, \quad \tau(N_1 + N_2) = \tau(N_1) + \tau(N_2), \tag{3.56a}$$

$$\forall N_1, N_2 \in \mathfrak{g}, \quad \tau([N_1, N_2]) = [\tau(N_1), \tau(N_2)]. \tag{3.56b}$$

For an anti-automorphism $\kappa : \mathfrak{g} \to \mathfrak{g}$, the second condition is replaced by the following:

$$\forall N_1, N_2 \in \mathfrak{g}, \quad \kappa([N_1, N_2]) = -[\kappa(N_1), \kappa(N_2)]. \tag{3.57}$$

Automorphisms and anti-automorphisms of Lie algebra $\mathfrak{g}$ induce automorphisms and anti-automorphisms of its Lie group $\mathcal{G}$, respectively. Arbitrary elements $\tau(N), \kappa(N) \in \mathfrak{g}$ define elements $\tau(G), \kappa(G) \in \mathcal{G}$ by relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau(G) = \tau(N)\tau(G), \quad \tau(G(0)) = \mathbb{1}, \tag{3.58a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa(G) = \kappa(N)\kappa(G), \quad \kappa(G(0)) = \mathbb{1}, \tag{3.58b}$$

respectively. An automorphism $\tau : \mathcal{G} \to \mathcal{G}$ of Lie group $\mathcal{G}$ satisfies the condition

$$\forall G_1, G_2 \in \mathcal{G}, \quad \tau(G_1 G_2) = \tau(G_1)\tau(G_2). \tag{3.59}$$

The analogous relation for an anti-automorphism $\kappa : \mathcal{G} \to \mathcal{G}$ has the form

$$\forall G_1, G_2 \in \mathcal{G}, \quad \kappa(G_1 G_2) = \kappa(G_2)\kappa(G_1). \tag{3.60}$$

Also if $\mathcal{M}$ has a structure of any associative algebra, we can use automorphisms and anti-automorphisms of this algebra.

As the last example we consider the matrix differential equation defined by the following action of the product $\mathfrak{g} \times \mathfrak{g}$ on $\mathcal{M}$:

$$\rho(N_1, N_2; L) = N_1 L - L N_2, \quad N_1, N_2 \in \mathfrak{g}. \tag{3.61}$$

It is the most general type of action that we consider in this paper and all previous examples considered are particular cases of this.

Integration of this action gives

$$\Phi(G_1, G_2; L) = G_1 L G_2, \quad G_1, G_2 \in \mathcal{G}, \tag{3.62}$$

where $G_1, G_2 \in \mathcal{G}$ corresponds to $N_1, N_2 \in \mathfrak{g}$, respectively. Matrix differential equation built by means of (3.61) has the form

$$\dot{L} = N_1(t, L)L - L N_2(t, L), \quad L(0) = L_0. \tag{3.63}$$

We can write its solution as

$$L(t) = G_1(t) L_0 G_2^{-1}(t), \tag{3.64}$$

where the flows $G_1(t)$ and $G_2(t)$ are solutions of the initial-value problems:

$$\dot{G}_1(t) = N_1(t, L)G_1(t), \quad G_1(0) = \mathbb{1}, \tag{3.65a}$$

$$\dot{G}_2(t) = N_2(t, L)G_2(t), \quad G_2(0) = \mathbb{1}. \tag{3.65b}$$

Without additional assumptions on N_1, N_2, we do not know how to find $\mathcal{G}$-invariant functions of Φ defined in (3.62). In the next two theorems, we write conditions which guarantee that determinant and singular values of $L(t)$ are first integrals. We recall that the singular values of L are the non-negative square roots of the eigenvalues of $L^{\dagger}L$ (in the case of complex matrices) or $L^{\mathrm{T}}L$ (in the case of real matrices).

Theorem 9. *If in* (3.63) *matrices N_1, N_2 are chosen in such a way that G_1 and G_2 defined in* (3.65a) *and* (3.65b) *satisfy the relation*

$$\det(G_1(t)G_2^{-1}(t)) = 1, \tag{3.66}$$

then the determinant of $L(t)$ is a first integral of (3.63).

Proof. From form (3.64) of $L(t)$ and elementary properties of the deteminant of matrix we obtain

$$\begin{aligned}\det(L(t)) &= \det(G_1(t))\det(L_0)\det(G_2^{-1}(t))\\ &= \det(G_1(t)G_2^{-1}(t))\det(L_0) = \det(L_0).\end{aligned} \tag{3.67}$$

The last equality is valid if and only if condition (3.66) is fulfilled. □

Condition (3.66) means that $G_1(t)$ and $G_2(t)$ have the same determinant. For example, it is the case when $G_1(t)$ and $G_2(t)$ belong to $\mathrm{SL}(n, \mathbb{K})$. Then N_1 and N_2 have to be traceless matrices.

The next theorem concerns matrix differential equations having singular values as first integrals.

Theorem 10. *If in* (3.63) *matrices N_1, N_2 belong to* $\mathrm{o}(n, \mathbb{R})$ *or* $\mathrm{u}(n, \mathbb{C})$, *then singular values of $L(t)$ are first integrals of* (3.63).

Proof (for the case of complex matrices). If we assume that N_1 and N_2 are skew-Hermitian, then $G_1(t)$ and $G_2(t)$ as solutions of (3.65a) and (3.65b) are unitary:

$$G_1^{\dagger}(t) = G_1^{-1}(t), \qquad G_2^{\dagger}(t) = G_2^{-1}(t). \tag{3.68}$$

From the above relations, we obtain

$$L^{\dagger}(t)L(t) = G_2(t)L_0^{\dagger}L_0G_2^{-1}(t) \tag{3.69}$$

and it is obvious that singular values of L are constants. □

Example 7. A Lie-admissible [13] generalization of the Heisenberg equation for observable A has the form

$$\dot{A} = -\frac{\mathrm{i}}{\hbar}(ARH_0 - H_0SA). \tag{3.70}$$

Here H_0 is a Hermitian matrix and matrices R and S satisfy the condition $R \neq \pm S$.

From the above considerations we see that if matrices R and S are Hermitian and commute with H_0, then singular values of A are constants.

We showed a number of matrix differential equations. These equations can appear in different branches of physics in a direct way (if objects in the theory have a matrix character) or in an indirect way (if matrix differential equations appear as matrix representations of dynamical systems). In the second case, we have many problems:

1. How to find a matrix representation?
2. Is this representation unique?
3. Is a level of difficulties during searches of different representations of analyzed system the same?

Regarding the first problem, we know systematic methods of mapping dynamical systems into matrix differential equations of required form only in a few cases. In the case of the Lax equation, they are dynamical systems with known tensor invariant of valence (1,1). It is possible to find such tensors, e.g. for bi-Hamiltonian systems. We also showed that is possible mapping in a systematic way the dynamical systems with a tensor invariant of valence (0,2) or (2,0) into the matrix equation with transposition. It is worth noticing that mapping the dynamical system into the matrix differential equation of the required form is not equivalent to finding the matrix representation of this type. Not always from the matrix equation can we obtain all differential equations of the analyzed dynamical system.

We mentioned that all Hamiltonian systems integrable in the Liouville sense have known Lax representations (in action–angle variables). In fact, for such systems we can construct in a systematic way all presented representations: with the complex conjugation, transposition, Hermitian conjugation, with two skew-symmetric or skew-Hermitian matrices [37], in the action–angle variables.

The Lax representations have also systems described by the projection method introduced by Olshanetsky and Perelomov [32] and this follows directly from constructions applied in this method.

For other systems in order to construct required matrix representation, we have to postulate a form of entries of matrices L and N (or L, N_1, N_2) as particular functions of dynamical variables $x^1, \dots, x^k$ with undeterminate coefficients. Using evolution equations for dynamical variables, we obtain the system of algebraic equations for undeterminate coefficients. This system does not always have a non-trivial solution, and furthermore, not always a non-trivial solution gives a representation which produce non-trivial first integrals. Moreover, as mentioned above, sometimes from obtained matrix differential equation, we cannot reconstruct all differential equations of the analyzed dynamical system.

Regarding the second problem, matrix representations of dynamical systems are not unique. In [27] it was shown that if a dynamical system has one Lax representation, then we can produce infinite sequence of the next Lax representation by means of Kronecker product. In fact, it is true also for all presented matrix representations [37]. If the system has one matrix representation of a certain type, then it has infinitely many representations of the same type. Additionally, sometimes the system has matrix representations of different types. We illustrate this statement with some examples.

Example 8. The equations of motion of the Toda lattice [42] of two particles in the Flaschka variables have the form [16]

$$\dot{b}_1 = -2a_1^2, \qquad \dot{b}_2 = 2a_1^2, \qquad \dot{a}_1 = a_1(b_1 - b_2). \tag{3.71}$$

This system has the well-known Lax representation:

$$\tilde{L} = \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \end{pmatrix}, \qquad \tilde{N} = \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \tag{3.72}$$

which gives two polynomial first integrals

$$I_1 = b_1 + b_2, \qquad I_2 = b_1^2 + b_2^2 + 2a_1^2. \tag{3.73}$$

For this system, we can calculate the matrix representation with the transposition

$$L = \begin{pmatrix} b_1 - \frac{3}{2}a_1 & 2b_1 - b_2 + 2a_1 \\ b_1 - 2b_2 + 2a_1 & 4b_2 + 6a_1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & \frac{1}{2}a_1 \\ -2a_1 & 0 \end{pmatrix}, \tag{3.74}$$

which also gives two first integrals.

Example 9. Generalized Halphen system [22] has the form

$$\begin{aligned} \dot{x}_1 &= a_1x_1^2 + (\lambda + a_1)[x_2x_3 - x_1(x_2 + x_3)], \\ \dot{x}_2 &= a_2x_2^2 + (\lambda + a_2)[x_3x_1 - x_2(x_3 + x_1)], \\ \dot{x}_3 &= a_3x_3^2 + (\lambda + a_3)[x_1x_2 - x_3(x_1 + x_2)], \end{aligned} \tag{3.75}$$

where (λ, a_1, a_2, a_3) are parameters. If $(\lambda, a_1, a_2, a_3) = (1, 0, 0, 0)$, then (3.75) coincides with the well-known Halphen system [21]. In [28] it was stated that if

$$\lambda = -\tfrac{1}{2}(a_1 + a_2 + a_3), \tag{3.76}$$

then the generalized Halphen system is integrable because it possesses two rational (degree zero) first integrals:

$$I_1 = \frac{x_1 - x_2}{x_2 - x_3}, \qquad I_2 = \frac{x_2 - x_3}{x_3 - x_1}. \tag{3.77}$$

Despite functional dependence of these first integrals,

$$I_2 = -\frac{1}{1 + I_1}, \tag{3.78}$$

the general conclusion about integrability remains valid. As the second first integral independent of I_1, we can choose the following rational function of degree 1:

$$I_3 = \frac{(a_1 - a_2 - a_3)x_2x_3 - (a_1 - a_2 + a_3)x_1x_3 - (a_1 + a_2 - a_3)x_1x_2}{x_1 - x_2}. \tag{3.79}$$

The generalized Halphen system has many matrix representations with transposition, e.g.

$$L = \begin{pmatrix} x_1 & -x_2 + x_3 \\ -2x_1 + x_2 & 0 \end{pmatrix},$$

$$N = \frac{1}{4}\begin{pmatrix} (a_1 + a_2 + a_3)x_1 & -(a_1 - a_2 - a_3)(x_1 - x_2) \\ 0 & (3a_1 - a_2 - a_3)x_1 + 4(a_2x_2 + a_3x_3) \end{pmatrix}. \tag{3.80}$$

The calculation of trace of $L(L^{-1})^{\mathrm{T}}$ gives the rational (degree 0) first integral which is the function of I_1:

$$\mathrm{Tr}(L(L^{-1})^{\mathrm{T}}) = 2 - \frac{4(x_1 - x_2)^2}{(x_2 - x_3)(-2x_1 + x_2 + x_3)} = 1 + 2I_1 + \frac{1}{1 + 2I_1}. \tag{3.81}$$

If we try to find the Lax representation of this system with entries linear in dynamical variables, then all these representations give only trivial first integrals. It is obvious because such Lax representation with linear entries gives only polynomial first integrals and the generalized Halphen system does not have any polynomial first integral. If we want to construct the Lax rapresentation with non-trivial first integrals we have to assume that entries of L and N depend rationally on dynamical variables. But then the number of equations determining entries of matrices L and N grows and calculations become more complicated.

Based on the last example, we see that sometimes it is easier to find the useful representation with transposition. By useful representation we mean a matrix representation giving non-trivial first integrals.

Now we try to answer the third question. We can conjecture that there are systems for which it is easier to find the Lax representation generating non-trivial first integrals. But there are systems for which it is easier to find a useful matrix representation of another type (e.g. with transposition). Sometimes representations of different types can give non-trivial first integrals (see Example 8). The formulation "a representation easier to find" means matrix representation with entries of L and N in a set of simpler functions of dynamical variables.

4. Hamiltonian matrix differential equations on a Lie algebra

In this section and the next section, we consider constructions of Hamiltonian subclasses of matrix differential equations defined by actions of Lie algebras. In this section we consider a case when $\mathcal{M} = \mathfrak{g}$, thus $\mathcal{M}$ is in particular a vector space. We assume that our action $\rho : \mathfrak{g} \times \mathcal{M} \to \mathcal{M}$ is a linear transformation also in the second argument. We call a linear action of a Lie algebra on a certain vector space a representation of this Lie algebra, and similarly we call a linear action of a Lie group a representation of this Lie group. In fact, all actions presented in examples (2.3) and (2.11) are linear in both arguments.

At first we introduce the next notions from representation theory. Let $\mathfrak{g}^\star$ denote the algebra dual to $\mathfrak{g}$. By definition, a co-representation $\Phi^\star : \mathcal{G} \times \mathfrak{g}^\star \to \mathfrak{g}^\star$ of Lie group $\mathcal{G}$ on $\mathfrak{g}^\star$ is the object dual to $\Phi : \mathcal{G} \times \mathfrak{g} \to \mathfrak{g}$, i.e.

$$\langle \Phi^\star(G, \xi), N\rangle \equiv \langle \Phi^\star_G(\xi), N\rangle := \langle \xi, \Phi_{G^{-1}}(N)\rangle, \tag{4.1}$$

where $G \in \mathcal{G}, \xi \in \mathfrak{g}^\star, N \in \mathfrak{g}$ and $\langle\cdot,\cdot\rangle : \mathfrak{g}^\star \times \mathfrak{g} \to \mathbb{R}$ denotes the duality map between $\mathfrak{g}^\star$ and $\mathfrak{g}$. Having co-representation $\Phi^\star$ of Lie group $\mathcal{G}$, we can calculate the appropriate co-representation $\rho^\star$ of Lie algebra $\mathfrak{g}$ analogously as for actions of Lie groups and Lie algebras:

$$\rho^\star(N, \xi) \equiv \rho^\star_N(\xi) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^\star_{\exp\{tN\}}(\xi), \tag{4.2}$$

where $N \in \mathfrak{g}, \xi \in \mathfrak{g}^\star$.

In order to construct Hamiltonian equations, we have to define a Poisson bracket and choose Hamiltonian functions. On $C^\infty(\mathfrak{g}^\star)$ there exists a natural Poisson bracket, the so-called Berezin–Kostant–Kirillov–Souriou bracket (BKKS bracket) [23]. Let $L \in \mathfrak{g}^\star$, $\varphi, \psi \in C^\infty(\mathfrak{g}^\star)$. Then this bracket has the form

$$\{\varphi, \psi\}(L) := \langle L, [\mathrm{d}\psi(L), \mathrm{d}\varphi(L)]\rangle, \tag{4.3}$$

where differentials $\mathrm{d}\varphi(L), \mathrm{d}\psi(L)$ belong to $(\mathfrak{g}^\star)^\star \cong \mathfrak{g}$.

As a Hamiltonian function we can use an arbitrary function φ from $C^\infty(\mathfrak{g}^\star)$. The BKKS bracket generates the following Hamiltonian vector field:

$$\dot{L} = \mathrm{ad}^\star_{\mathrm{d}\varphi(L)}(L). \tag{4.4}$$

If function φ belongs to the set of $\mathcal{G}$-invariant functions of the co-adjoint representation $\mathrm{Ad}^\star$, i.e.

$$\forall\, G \in \mathcal{G}, \quad \varphi(\mathrm{Ad}^\star_G(L)) = \varphi(L), \tag{4.5}$$

then we obtain trivial dynamics. It is obvious that from an infinitesimal version of the above invariance condition,

$$\forall\, N \in \mathfrak{g}, \quad \langle N, \mathrm{ad}^\star_{\mathrm{d}\varphi(L)}(L)\rangle = 0. \tag{4.6}$$

As a consequence, $\mathcal{G}$-invariant functions of co-adjoint representation are Casimir functions of BKKS bracket.

In order to generate non-trivial dynamics with $\mathcal{G}$-invariant Hamiltonian, it is necessary in the definition of BKKS bracket to replace the commutator by another Lie bracket in $\mathfrak{g}$. This bracket can be built by means of a certain linear operator $R \in \mathrm{End}(\mathfrak{g})$ in the following way:

$$\forall\, N_1, N_2 \in \mathfrak{g}, \quad [N_1, N_2]_R := [R(N_1), N_2] + [N_1, R(N_2)]. \tag{4.7}$$

A skew-symmetry and linearity of $[\cdot,\cdot]_R$ is obvious from its definition. From Jacobi identity, we obtain the following condition [39]:

$$\begin{aligned}&\forall N_1, N_2, N_3 \in \mathfrak{g},\\ &[B_R(N_1,N_2),N_3]+[B_R(N_2,N_3),N_1]+[B_R(N_3,N_1),N_2]=0,\end{aligned} \tag{4.8}$$

where $B_R(N_1, N_2)$ is defined by

$$B_R(N_1,N_2)=[R(N_1),R(N_2)]-R([N_1,N_2]_R). \tag{4.9}$$

Operators which satisfy this relation are called classical R matrices. Condition (4.8) is very complicated and usually two types of sufficient conditions are analyzed: the classical Yang–Baxter identity, in short (CYBE)

$$B_R(N_1,N_2)=0, \tag{4.10}$$

and the modified Yang–Baxter identity, in short (mCYBE)

$$B_R(N_1,N_2)=\alpha[N_1,N_2], \tag{4.11}$$

where α is a real parameter. The BKKS bracket with $[\cdot,\cdot]_R$

$$\{\varphi,\psi\}_R(L):=\langle L,[\mathrm{d}\psi(L),\mathrm{d}\varphi(L)]_R\rangle, \tag{4.12}$$

and any $\mathcal{G}$-invariant Hamiltonian function φ gives the following equation on $\mathfrak{g}^\star$:

$$\dot{L}=\mathrm{ad}^\star_{R(\mathrm{d}\varphi(L))}(L). \tag{4.13}$$

If $\mathfrak{g}$ admits a non-degenerate invariant bilinear form $(\cdot,\cdot)$, then this equation takes Lax's form

$$\dot{L}=[R(\mathrm{grad}\,\varphi(L)),L]. \tag{4.14}$$

The invariance condition means that for any $N_1, N_2, N_3 \in \mathfrak{g}$,

$$([N_1,N_2],N_3)+(N_2,[N_1,N_3])=0. \tag{4.15}$$

The existence of this invariant form makes it possible to identify elements from $\mathfrak{g}^\star$ with elements from $\mathfrak{g}$, the co-adjoint representation with the adjoint representation, differentials of functions with gradients and $\mathcal{G}$-invariant functions of the co-adjoint representation with $\mathcal{G}$-invariant functions of the adjoint representation. An example of such a form is the Killing form for semisimple Lie algebras.

Additionally BKKS bracket with $[\cdot,\cdot]_R$ for any two $\mathcal{G}$-invariant functions of adjoint representation is equal to zero.

Following work of Semenov-Tian-Shansky [39], we want to build Hamiltonian equations of the form

$$\dot{L}=\rho_N(L), \tag{4.16}$$

where ρ is an arbitrary representation of $\mathfrak{g}$ on $\mathfrak{g}$. A certain solution of this task was proposed by Bordemann [7]. We use some ideas of Bordemman but our analysis is different (closer to the oryginal work of Semenov-Tian-Shansky).

At first we construct Hamiltonian equations on $\mathfrak{g}^\star$:

$$\dot{L} = \rho^\star_N(L), \quad L \in \mathfrak{g}^\star, \tag{4.17}$$

where $\rho^\star$ is a co-representation of $\mathfrak{g}^\star$. We need to define a new Poisson bracket on $C^\infty(\mathfrak{g}^\star)$. We have two possibilities:

- we can construct a completely new Poisson bracket different from BKKS bracket depending on the form of ρ,
- do not change the general form of BKKS bracket but replace the commutator appearing in (4.3) by another map $[\cdot,\cdot]_\rho : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ depending on a form of ρ and satisfying all conditions determining a Lie bracket.

We apply the second approach and we postulate the following form of this new Lie bracket:

$$[N_1, N_2]_\rho = \rho_{N_1}(N_2) - \rho_{N_2}(N_1). \tag{4.18}$$

A skew-symmetry and linearity of this Lie bracket in $\mathfrak{g}$ is obvious from its definition. In order to analyze Jacobi identity, we obtain some useful relation for representation ρ. We recall the first condition in the definition of an action of a Lie algebra on a manifold

$$\forall\, N_1, N_2 \in \mathfrak{g}, \quad \rho_{[N_1,N_2]} = |[\rho_{N_1}, \rho_{N_2}]|. \tag{4.19}$$

In the case of representation, this condition transforms into

$$\forall\, L, N_1, N_2 \in \mathfrak{g}, \quad \rho_{[N_1,N_2]}(L) = \rho_{N_1}(\rho_{N_2}(L)) - \rho_{N_2}(\rho_{N_1}(L)). \tag{4.20}$$

From Jacobi identity by means of (4.20), we obtain one condition restricting the form of representation ρ:

$$\sum_{(i,j,k)\in S} \rho_{\{\rho_{N_i}(N_j) - \rho_{N_j}(N_i)\}}(N_k) + \rho_{[N_j,N_i]}(N_k) = 0, \tag{4.21}$$

where S is a set of cyclic permutations of $\{1, 2, 3\}$.

Now we can check the form of Hamiltonian equations generated by this new bracket

$$\{\varphi, \psi\}_\rho(L) := \langle L, [\mathrm{d}\psi(L), \mathrm{d}\varphi(L)]_\rho\rangle. \tag{4.22}$$

We choose an arbitrary $\mathcal{G}$-invariant function φ of the co-representation $\Phi^\star$:

$$\forall\, G \in \mathcal{G}, \quad \varphi(\Phi^\star_G(L)) = \varphi(L), \tag{4.23}$$

as a Hamiltonian. An infinitesimal version of this condition has the form

$$\forall\, N \in \mathfrak{g}, \quad 0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi(\Phi^\star_{\exp\{tN\}}(L)) = \langle \mathrm{d}\varphi, \rho^\star_N(L)\rangle. \tag{4.24}$$

Using the relation between $\Phi^\star$ and Φ given in (4.1), we can obtain an appropriate relation between $\rho^\star$ and ρ:

$$\langle \rho^*_{N_1}(\xi), N_2\rangle = -\langle \xi, \rho_{N_1}(N_2)\rangle. \tag{4.25}$$

By means of the above relation, we can rewrite condition (4.24) as

$$\forall\, N \in \mathfrak{g}, \quad \langle L, \rho_N(\mathrm{d}\varphi)\rangle = 0. \tag{4.26}$$

We calculate the Poisson bracket $\{\varphi, \psi\}_\rho$ with an arbitrary $\mathcal{G}$-invariant function φ of co-representation $\Phi^\star$ as a Hamiltonian:

$$\begin{aligned}\{\varphi, \psi\}_\rho &= \langle L, [\mathrm{d}\psi, \mathrm{d}\varphi]_\rho\rangle = \langle L, \rho_{\mathrm{d}\psi}(\mathrm{d}\varphi)\rangle - \langle L, \rho_{\mathrm{d}\varphi}(\mathrm{d}\psi)\rangle \\ &= -\langle L, \rho_{\mathrm{d}\varphi}(\mathrm{d}\psi)\rangle = \langle \rho^*_{\mathrm{d}\varphi}(L), \mathrm{d}\psi\rangle.\end{aligned} \tag{4.27}$$

As a result, we obtain a Hamiltonian equation of the form (4.17) with $N = \mathrm{d}\varphi$.

It is worth noticing that if representation ρ is symmetric or skew-symmetric, i.e.

$$\forall\, N_1, N_2 \in \mathfrak{g}, \quad \rho_{N_1}(N_2) = \pm\rho_{N_2}(N_1), \tag{4.28}$$

then Poisson bracket $\{\cdot,\cdot\}_\rho$ with the $\mathcal{G}$-invariant Hamiltonian φ generates only trivial dynamics. In a symmetric case the Lie bracket $[\cdot,\cdot]_\rho$ is identically equal to zero and in a skew-symmetric case the triviality of dynamics follows from $\mathcal{G}$-invariant character of Hamiltonian. We can recapitulate obtained results in the following theorem.

Theorem 11. *If a representation $\rho : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfies* (4.21), *then a Hamiltonian equation on $\mathfrak{g}^\star$ generated by Poisson bracket $\{\cdot,\cdot\}_\rho$ with the $\mathcal{G}$-invariant with respect to $\Phi^\star$ Hamiltonian φ has the form* (4.17). *Moreover, any two $\mathcal{G}$-invariant functions are pairwise in involution with respect to this bracket*.

Proof. The first part of the theorem is obvious from the above considerations. The second part is easy to check the following calculations (4.27) and using $\mathcal{G}$-invariance of φ. □

If the analyzed representation does not satisfy relation (4.21) or is symmetric (skew-symmetric), then we can try to build another Lie bracket (and as a consequence a Poisson bracket) using the counterpart of a classical R matrix. We denote it by r and call a classical r matrix. By definition, a classical r matrix is a linear operator $r \in \mathrm{End}(\mathfrak{g})$ which generates in $\mathfrak{g}$ a structure of Lie algebra by means of the bracket

$$\forall\, N_1, N_2 \in \mathfrak{g}, \quad [N_1, N_2]_B = \rho_{r(N_1)}(N_2) - \rho_{r(N_2)}(N_1). \tag{4.29}$$

We call this bracket the Bordemann bracket because it has a structure similar to the bracket from his work [7]. The linearity and skew-symmetry is obvious by definition. Now we pass to Jacobi identity. Its particular terms are the following:

$$\begin{aligned}
[[N_1, N_2]_B, N_3]_B &= \rho_{r([N_1,N_2]_B)}(N_3) - \rho_{r(N_3)}([N_1, N_2]_B) \\
&= \rho_{r(\rho_{r(N_1)}(N_2))}(N_3) - \rho_{r(\rho_{r(N_2)}(N_1))}(N_3) - \rho_{r(N_3)}(\rho_{r(N_1)}(N_2)) \\
&\quad + \rho_{r(N_3)}(\rho_{r(N_2)}(N_1)), \\
[[N_2, N_3]_B, N_1]_B &= \rho_{r([N_2,N_3]_B)}(N_1) - \rho_{r(N_1)}([N_2, N_3]_B) \\
&= \rho_{r(\rho_{r(N_2)}(N_3))}(N_1) - \rho_{r(\rho_{r(N_3)}(N_2))}(N_1) - \rho_{r(N_1)}(\rho_{r(N_2)}(N_3)) \\
&\quad + \rho_{r(N_1)}(\rho_{r(N_3)}(N_2)), \\
[[N_3, N_1]_B, N_2]_B &= \rho_{r([N_3,N_1]_B)}(N_2) - \rho_{r(N_2)}([N_3, N_1]_B) \\
&= \rho_{r(\rho_{r(N_3)}(N_1))}(N_2) - \rho_{r(\rho_{r(N_1)}(N_3))}(N_2) - \rho_{r(N_2)}(\rho_{r(N_3)}(N_1)) \\
&\quad + \rho_{r(N_2)}(\rho_{r(N_1)}(N_3)).
\end{aligned} \tag{4.30}$$

If we define quantity B_r as

$$\begin{aligned}
B_r(N_1, N_2) &:= [r(N_1), r(N_2)] - r([N_1, N_2]_B) \\
&= [r(N_1), r(N_2)] - r(\rho_{r(N_1)}(N_2)) - r(\rho_{r(N_2)}(N_1)),
\end{aligned} \tag{4.31}$$

and calculate $-\rho_{B_r(N_1,N_2)}(N_3)$ using representation identity (4.20), then we obtain

$$\begin{aligned}
-\rho_{B_r(N_1,N_2)}(N_3) &= -\rho_{[r(N_1),r(N_2)]}(N_3) + \rho_{r(\rho_{r(N_1)}(N_2))}(N_3) - \rho_{r(\rho_{r(N_2)}(N_1))}(N_3) \\
&= -\rho_{r(N_1)}(\rho_{r(N_2)}(N_3)) + \rho_{r(N_2)}(\rho_{r(N_1)}(N_3)) + \rho_{r(\rho_{r(N_1)}(N_2))}(N_3) \\
&\quad -\rho_{r(\rho_{r(N_2)}(N_1))}(N_3).
\end{aligned} \tag{4.32}$$

Looking at the obtained expression, we recognize four terms from the right-hand sides of (4.30). If we repeat calculations for $-\rho_{B_r(N_2,N_3)}(N_1)$ and $-\rho_{B_r(N_3,N_1)}(N_2)$, then we conclude that it is possible to write Jacobi identity as

$$\rho_{B_r(N_1,N_2)}(N_3) + \rho_{B_r(N_2,N_3)}(N_1) + \rho_{B_r(N_3,N_1)}(N_2) = 0. \tag{4.33}$$

As a consequence the bracket $[\cdot, \cdot]_B$ defined in (4.29) is a Lie bracket in $\mathfrak{g}$ if and only if for any $N_1, N_2, N_3 \in \mathfrak{g}$, the condition (4.33) is satisfied. We can construct two sufficient conditions, counterparts of (CYBE) and (mCYBE)

$$\forall N_1, N_2 \in \mathfrak{g}, \quad B_r(N_1, N_2) = 0, \tag{4.34a}$$

$$\forall N_1, N_2 \in \mathfrak{g}, \quad B_r(N_1, N_2) = [D(N_1), D(N_2)], \tag{4.34b}$$

where on mapping $D : \mathfrak{g} \to \mathfrak{g}$ we have to put such conditions which guarantee that (4.33) is satisfied identically.

Theorem 12. *If for representation* ρ, *we find a map D which satisfies two conditions*:

$$\rho_{D(N_1)}(N_2) = -\rho_{D(N_2)}(N_1), \tag{4.35a}$$

$$\rho_{D(N_1)}(\rho_{D(N_2)}(N_3)) + \rho_{D(N_2)}(\rho_{D(N_3)}(N_1)) + \rho_{D(N_3)}(\rho_{D(N_1)}(N_2)) = 0, \tag{4.35b}$$

or one condition

$$\rho_{D(N_1)}(N_2)=\rho_{D(N_2)}(N_1) \tag{4.36}$$

then equality (4.33) *is satisfied identically.*

Proof. We insert (4.34b) into (4.33) and use (4.20). We obtain

$$\begin{aligned}&\rho_{[D(N_1),D(N_2)]}(N_3)+\rho_{[D(N_2),D(N_3)]}(N_1)+\rho_{[D(N_3),D(N_1)]}(N_2)\\&\quad=\rho_{D(N_1)}(\rho_{D(N_2)}(N_3))-\rho_{D(N_2)}(\rho_{D(N_1)}(N_3))+\rho_{D(N_2)}(\rho_{D(N_3)}(N_1))\\&\qquad-\rho_{D(N_3)}(\rho_{D(N_2)}(N_1))+\rho_{D(N_3)}(\rho_{D(N_1)}(N_2))-\rho_{D(N_1)}(\rho_{D(N_3)}(N_2)).\end{aligned} \tag{4.37}$$

If D satisfies (4.36), then the first term reduces with sixth, second with third, fourth with fifth and we obtain 0. If D satisfies (4.35a), then the right-hand side of (4.37) transforms into

$$2\{\rho_{D(N_1)}(\rho_{D(N_2)}(N_3))+\rho_{D(N_2)}(\rho_{D(N_3)}(N_1))+\rho_{D(N_3)}(\rho_{D(N_1)}(N_2))\}, \tag{4.38}$$

which is equal to zero when (4.35b) is fulfilled. □

A first class of conditions on D, i.e. (4.35a) and (4.35b) corresponds to Bordemann results [7]. But we obtain also the second one given by only one equation (4.36).

If for our representation ρ of $\mathfrak{g}$ on $\mathfrak{g}$ we find a map D, which fulfills (4.35a) and (4.35b) or (4.36), and next we find a classical r matrix satisfying (4.34b) with the calculated form of D, then we can construct Hamiltonian equations on $\mathfrak{g}^\star$. As a Poisson bracket we take

$$\{\varphi,\psi\}_B(L)=\langle L,[\mathrm{d}\psi,\mathrm{d}\varphi]_B\rangle, \tag{4.39}$$

where $[\cdot,\cdot]_B$ is defined in (4.29), and as a Hamiltonian an arbitrary $\mathcal{G}$-invariant function φ of co-representation $\Phi^\star$. Calculations of the above bracket yield

$$\begin{aligned}\langle L,[\mathrm{d}\psi,\mathrm{d}\varphi]_B\rangle&=\langle L,\rho_{r(\mathrm{d}\psi)}(\mathrm{d}\varphi)\rangle-\langle L,\rho_{r(\mathrm{d}\varphi)}(\mathrm{d}\psi)\rangle=-\langle L,\rho_{r(\mathrm{d}\varphi)}(\mathrm{d}\psi)\rangle\\&=\langle \mathrm{d}\psi,\rho^\star_{r(\mathrm{d}\varphi)}(L)\rangle,\end{aligned} \tag{4.40}$$

and we conclude that a Hamiltonian equation has the form (4.17) with $N=r(\mathrm{d}\varphi)$.

From above calculation it is obvious that the Poisson bracket (4.39) of any two $\mathcal{G}$-invariant functions $\Phi^\star$ is equal to zero.

We showed two constructions of Hamiltonian equations of required form (4.17) on $\mathfrak{g}^\star$. But we want to obtain appropriate Hamiltonian equations on $\mathfrak{g}$. In order to make this, we need a scalar product invariant with respect to ρ:

$$\forall N_1,N_2,N_3\in\mathfrak{g},\quad (\rho_{N_1}(N_2),N_3)+(N_2,\rho_{N_1}(N_3))=0. \tag{4.41}$$

This product makes it possible to identify elements from $\mathfrak{g}^\star$ with elements with $\mathfrak{g}$, co-representation $\rho^\star$ with representation ρ, differentials of functions with gradients, and $\mathcal{G}$-invariant functions of $\Phi^\star$ with $\mathcal{G}$-invariant functions of Φ. Now we repeat calculations of

Poisson brackets with Lie bracket $[\cdot,\cdot]_\rho$ and $[\cdot,\cdot]_B$ presented in (4.27) and (4.40), respectively, using the above-mentioned identifications. As a Hamiltonian function, we take an arbitrary $\mathcal{G}$-invariant with respect to Φ function φ. Now an infinitesimal version of the invariance condition has the form

$$\forall\, N \in \mathfrak{g}, \quad (\operatorname{grad}\varphi, \rho_N(L)) = 0. \tag{4.42}$$

Using the above condition, we obtain

$$\begin{aligned}
\langle L, [\mathrm{d}\varphi, \mathrm{d}\psi]_\rho\rangle &= (L, \rho_{\operatorname{grad}\psi}(\operatorname{grad}\varphi)) - (L, \rho_{\operatorname{grad}\varphi}(\operatorname{grad}\psi))\\
&= -(\rho_{\operatorname{grad}\psi}(L), \operatorname{grad}\varphi) + (\rho_{\operatorname{grad}\varphi}(L), \operatorname{grad}\psi)\\
&= (\rho_{\operatorname{grad}\varphi}(L), \operatorname{grad}\psi),\\
\langle L, [\mathrm{d}\varphi, \mathrm{d}\psi]_B\rangle &= (L, \rho_{r(\operatorname{grad}\psi)}(\operatorname{grad}\varphi)) - (L, \rho_{r(\operatorname{grad}\varphi)}(\operatorname{grad}\psi))\\
&= -(\rho_{r(\operatorname{grad}\psi)}(L), \operatorname{grad}\varphi) + (\rho_{r(\operatorname{grad}\varphi)}(L), \operatorname{grad}\psi)\\
&= (\rho_{r(\operatorname{grad}\varphi)}(L), \operatorname{grad}\psi).
\end{aligned} \tag{4.43}$$

In this way, we obtain two families of Hamiltonian equations of the required form on $\mathfrak{g}$:

$$\dot{L} = \rho_{\operatorname{grad}\varphi(L)}(L), \tag{4.44a}$$

$$\dot{L} = \rho_{r(\operatorname{grad}\varphi(L))}(L), \tag{4.44b}$$

respectively.

Recapitulating, we have two constructions of Hamiltonian equations of the form (4.16). If representation ρ satisfies condition (4.21), then we can construct Hamiltonian equation (4.44a). For symmetric or skew-symmetric representations $\rho : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ this construction gives only trivial results. The second construction consists of two steps: at first we have to find a map D, and then find a classical matrix r. In both constructions we need a scalar product invariant with respect to ρ. Till now all necessary elements of this construction have been known only for the adjoint representation. This representation is skew-symmetric and only the second construction gives non-trivial results. In this case $D(N) = N$, the counterpart of (mCYBE) transforms exactly in (mCYBE), and if we take as $\mathfrak{g}$ any semisimple Lie algebra, then we have an invariant scalar product, namely the Killing form.

5. Hamiltonian matrix differential equations on a Lie group

In this section we construct Hamiltonian matrix differential equations of the forms analyzed in the previous sections on Lie group $\mathcal{G}$. We assume that $\mathcal{G}$ is such a Lie group that its Lie algebra is equipped with a non-degenerate scalar product $(\cdot,\cdot)$ invariant with respect to the adjoint action

$$\forall\, N_1, N_2, N_3 \in \mathfrak{g}, \quad ([N_1, N_2], N_3) + (N_2, [N_1, N_3]) = 0. \tag{5.1}$$

The above condition is an infinitesimal version of

$$\forall\, G \in \mathcal{G}, \quad (\mathrm{Ad}_G(N_2), \mathrm{Ad}_G(N_3)) = (N_2, N_3). \tag{5.2}$$

We construct separately Hamiltonian equations which have the form of the Lax equation and the forms of matrix differential equations defined by an arbitrary automorphism and anti-automorphism of Lie algebra $\mathfrak{g}$. We make this using one type of Poisson bracket and choosing as Hamiltonians $\mathcal{G}$-invariant functions with respect to appropriate actions.

These construction for the Lax equation and the equation with an arbitrary automorphism are known from works of Semenov-Tian-Shansky [39,40]. We present them here for completeness and because in original works proofs are not explicitly presented or they are made using advanced methods.

For any $\varphi \in C^\infty(\mathcal{G})$, we define left and right gradients $D\psi$, $D'\psi \in \mathfrak{g}$, respectively, as

$$(D\psi(L), N) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi(\mathrm{e}^{tN}L), \tag{5.3a}$$

$$(D'\psi(L), N) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi(L\,e^{tN}), \tag{5.3b}$$

where $N \in \mathfrak{g}$. From definitions it is obvious that for any $\psi \in C^\infty(\mathcal{G})$, gradients $D\psi$ and $D'\psi$ are related by equality

$$D'\psi(L) = \mathrm{Ad}_{L^{-1}}(D\psi(L)). \tag{5.4}$$

In order to construct a Poisson structure we need a unitary classical R matrix, (which satisfies the standard (mCYBE)). The unitarity means that $R \in \mathrm{End}(\mathfrak{g})$ satisfies

$$\forall N_1, N_2 \in \mathfrak{g}, \quad (R(N_1), N_2) + (N_1, R(N_2)) = 0. \tag{5.5}$$

By means of left and right gradients, and the unitary classical R matrix, we can construct the Poisson bracket for any $\varphi, \psi \in C^\infty(\mathcal{G})$ in the following way:

$$\{\varphi, \psi\}_S(L) := \tfrac{1}{2}(R(D\varphi(L)), D\psi(L)) - \tfrac{1}{2}(R(D'\varphi(L)), D'\psi(L)). \tag{5.6}$$

It was introduced by Sklyanin [41]. The skew-symmetry and linearity is obvious from its definition and the proof of Jacobi identity can be found in Semenov-Tian-Shansky [40]. This bracket is well defined also on associative algebras.

Now we find the Hamiltonian vector field generated by this Poisson bracket and Hamiltonian $\varphi \in C^\infty(\mathcal{G})$. Using (5.4), we can rewrite (5.6) as

$$\{\varphi, \psi\}_S(L) = \tfrac{1}{2}(\underbrace{R(D\varphi(L)) - \mathrm{Ad}_L(R(D'\varphi(L)))}, D\psi(L)). \tag{5.7}$$

The marked expression belongs to the tangent space to $\mathcal{G}$ at identity element of $\mathcal{G}$. We have to translate it to point $L \in \mathcal{G}$. We can do it by means of differentials of right or left translations on $\mathcal{G}$. Appropriate Hamiltonian equations generated by translated vector fields have forms

$$\dot{L} = \tfrac{1}{2}R(D\varphi(L))L - \tfrac{1}{2}LR(D'\varphi(L)), \tag{5.8a}$$

$$\dot{L} = \tfrac{1}{2}LR(D\varphi(L)) - \tfrac{1}{2}L^2R(D'\varphi(L))L^{-1}, \tag{5.8b}$$

respectively. In further considerations we will analyze only (5.8a) because only it gives matrix differential equations related to particular Lie algebra actions analyzed in Section 3.

Let φ be an arbitrary $\mathcal{G}$-invariant function of adjoint representation of $\mathcal{G}$, i.e.

$$\varphi(\mathrm{Ad}_G(L)) = \varphi(L), \tag{5.9}$$

We denote the set of such functions by $\mathcal{I}_{\mathrm{Ad}}(\mathcal{G})$. From the infinitesimal version of the above condition

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi(\mathrm{e}^{tN} L\, \mathrm{e}^{-tN}) = (D\varphi(L), N) - (D'\varphi(L), N), \tag{5.10}$$

we obtain the relation between the left and the right gradient of $\varphi \in \mathcal{I}_{\mathrm{Ad}}(\mathcal{G})$:

$$D'\varphi(L) = D\varphi(L). \tag{5.11}$$

Additionally, it is easy to show that

$$D\varphi(\mathrm{Ad}_G(L_0)) = \mathrm{Ad}_G(D\varphi(L_0)) \tag{5.12}$$

for any $G \in \mathcal{G}$ and $\varphi \in \mathcal{I}_{\mathrm{Ad}}(\mathcal{G})$.

By means of the classical R matrix, we define two new mappings

$$R_+ = \tfrac{1}{2}(R + \mathbb{1}), \qquad R_- = \tfrac{1}{2}(R - \mathbb{1}). \tag{5.13}$$

Using the above relations we can prove the following theorem.

Theorem 13. *Let $\varphi \in \mathcal{I}_{\mathrm{Ad}}(\mathcal{G})$ and Lie algebra $\mathfrak{g}$ be equipped with non-degenerate, invariant (with respect to* ad*) scalar product. Let $R \in \mathrm{End}(\mathfrak{g})$ be a unitary classical R matrix. Then,*

1. *the Hamiltonian equation* (5.8a) *transforms into the Lax equation*

$$\dot{L} = N(L)L - LN(L), \quad N(L) = \tfrac{1}{2}R(D\varphi(L)), \tag{5.14}$$

2. *any two functions from $\mathcal{I}_{\mathrm{Ad}}(\mathcal{G})$ are in involution with respect to the Sklyanin bracket* (5.6),
3. *each equation*

$$\dot{L} = N_+(L)L - LN_+(L), \quad N_+(L) = R_+(D\varphi(L)), \tag{5.15a}$$

$$\dot{L} = N_-(L)L - LN_-(L), \quad N_-(L) = R_-(D\varphi(L)), \tag{5.15b}$$

is equivalent to (5.14),

4. *the solution of* (5.14) *with the initial condition $L(0) = L_0$ can be written in two forms*:

$$L(t) = G_+(t)L_0G_+^{-1}(t) = G_-(t)L_0G_-^{-1}(t), \tag{5.16}$$

where flows $F(t)$ and $G(t)$ are solutions of the initial-value problems:

$$\dot{G}_+ = N_+(L)G_+, \quad G_+(0) = \mathbb{1}, \tag{5.17a}$$

$$\dot{G}_- = N_-(L)G_-, \quad G_-(0) = \mathbb{1}. \tag{5.17b}$$

Moreover, these flows are related to the solution $L(t)$ *of* (5.14) *with the initial condition* $L(0) = L_0$ *in the following way:*

$$G_-^{-1}(t)G_+(t) = \mathrm{e}^{tD\varphi(L_0)}, \tag{5.18a}$$

$$G_+(t)\,\mathrm{e}^{D\varphi(L_0)}G_+^{-1}(t) = G_-(t)\,\mathrm{e}^{D\varphi(L_0)}G_-^{-1}(t) = \mathrm{e}^{D\varphi(L(t))}. \tag{5.18b}$$

Proof.

1. It is sufficient to apply (5.11) to (5.8a).
2. For any two functions $\varphi_1, \varphi_2 \in \mathcal{I}_{\mathrm{Ad}}(\mathcal{G})$, we obtain

$$\begin{aligned}\{\varphi_1, \varphi_2\}_S &= \tfrac{1}{2}(R(D\varphi_1), D\varphi_2) - \tfrac{1}{2}(R(D'\varphi_1), D'\varphi_2)\\ &= \tfrac{1}{2}(R(D\varphi_1), D\varphi_2) - \tfrac{1}{2}(R(D'\varphi_1), D\varphi_2)\\ &= -\tfrac{1}{2}(D\varphi_1, R(D\varphi_2)) + \tfrac{1}{2}(D'\varphi_1, R(D\varphi_2)) = 0.\end{aligned}$$

3. To prove the equivalence (5.15a) with (5.14), we add to the right-hand side of Poisson bracket (5.6) two terms which do not change this bracket

$$\tfrac{1}{2}(D\varphi, D\psi) - \tfrac{1}{2}(D'\varphi, D\psi) = 0, \qquad \tfrac{1}{2}(D\varphi, D'\psi) - \tfrac{1}{2}(D'\varphi, D'\psi) = 0. \tag{5.19}$$

We obtain

$$\begin{aligned}\{\varphi, \psi\}_S(L) &= \tfrac{1}{2}(R(D\varphi(L)), D\psi(L)) - \tfrac{1}{2}(R(D'\varphi(L)), D'\psi(L))\\ &\quad +\tfrac{1}{2}(D\varphi(L), D\psi(L)) - \tfrac{1}{2}(D'\varphi(L), D\psi(L))\\ &\quad -\tfrac{1}{2}(D'\varphi(L), D'\psi(L)) + \tfrac{1}{2}(D\varphi(L), D'\psi(L))\\ &= (R_+(D\varphi(L)), D\psi(L)) - (R_+(D'\varphi(L)), D'\psi(L))\\ &\quad -\tfrac{1}{2}(D'\varphi(L), D\psi(L)) + \tfrac{1}{2}(D\varphi(L), D'\psi(L))\\ &= (R_+(D\varphi(L)), D\psi(L)) - (R_+(D'\varphi(L)), D'\psi(L))\\ &\quad -\tfrac{1}{2}(D\varphi(L), D\psi(L)) + \tfrac{1}{2}(D\varphi(L), D'\psi(L))\\ &= (R_+(D\varphi(L)), D\psi(L)) - (R_+(D'\varphi(L)), D'\psi(L)),\end{aligned}$$

from which follows (5.15a). Analogously, we obtain the equivalence of (5.15b) with (5.14). Additionally, adding (5.15a) to (5.15b) and using definitions of R_- and R_+ we obtain (5.14).

4. The existence of two forms of solutions of (5.14) is obvious from the equivalence of (5.14) with (5.15a) and (5.15b). In order to prove (5.18a) we observe that $\mathrm{e}^{tD\varphi(L_0)}$ satisfies the initial-value problem

$$\dot{Y}(t) = D\varphi(L_0)Y(t), \quad Y(0) = \mathbb{1}. \tag{5.20}$$

Let $Z(t) = G_-^{-1}(t)G_+(t)$. Then $Z(0) = \mathbb{1}$. We differentiate Z with respect to t:

$$\begin{aligned}\frac{\mathrm{d}Z(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(G_-^{-1}(t)G_+(t))\\ &= G_-^{-1}(t)(-N_- + N_+)G_+(t) = G_-^{-1}(t)D\varphi(L(t))G_-(t)G_-^{-1}(t)G_+(t)\\ &= D\varphi(L_0)G_-^{-1}(t)G_+(t) = D\varphi(L_0)Z(t).\end{aligned} \tag{5.21}$$

By the uniqueness theorem for initial-value problems, it follows that $Z(t) = \mathrm{e}^{tD\varphi(L_0)}$. In order to obtain (5.18b), we apply (5.12) and (5.16) to $\mathrm{e}^{tD\varphi(L_0)}$. □

Now we find a form of Hamiltonian equation generated by Poisson bracket (5.6) with an arbitrary $\mathcal{G}$-invariant function φ of the action $\Phi^{\tau}_{G}(L) = \tau(G)LG^{-1}$ as a Hamiltonian. Here τ denotes an arbitrary automorphism of $\mathfrak{g}$. We recall that an automorphism of a Lie algebra induces an appropriate automorphism of its Lie group.

We denote by $\mathcal{I}_{\tau}(\mathcal{G})$ the set of $\mathcal{G}$-invariant functions of Φ^{τ}. We assume that automorphism τ is orthogonal with respect to the scalar product in $\mathfrak{g}$, i.e.

$$(\tau(N_1), \tau(N_2)) = (N_1, N_2). \tag{5.22}$$

Using the infinitesimal version of the $\mathcal{G}$-invariance condition and orthogonality of the scalar product, we obtain the relation between $D\varphi$ and $D'\varphi$:

$$D\varphi(L) = \tau(D'\varphi(L)). \tag{5.23}$$

It is easy to check that the following equality

$$\forall\, L_0, G \in \mathcal{G}, \quad D'\varphi(\tau(G)L_0G^{-1}) = \mathrm{Ad}_G(D'\varphi(L_0)), \tag{5.24}$$

is valid for any $\varphi \in \mathcal{I}_{\tau}(\mathcal{G})$. Now we are able to prove the following theorem.

Theorem 14. *Let $\varphi \in \mathcal{I}_{\tau}(\mathcal{G})$ and Lie algebra $\mathfrak{g}$ be equipped with a non-degenerate invariant scalar product. Additionally, suppose that orthogonal automorphism τ of $\mathfrak{g}$ commutes with a unitary classical R matrix. Then*

1. *Hamiltonian equation* (5.8a) *has the form*

$$\dot{L} = \tau(N(L))L - LN, \quad N(L) = \tfrac{1}{2}R(D'\varphi(L)), \tag{5.25}$$

2. *any two functions $\varphi_1, \varphi_2 \in \mathcal{I}_{\tau}(\mathcal{G})$ are in involution with respect to the Sklyanin bracket* (5.6),
3. *each equation*

$$\dot{L} = \tau(N_+(L))L - LN_+, \quad N_+(L) = R_+(D'\varphi(L)), \tag{5.26a}$$

$$\dot{L} = \tau(N_-(L))L - LN_-, \quad N_-(L) = R_-(D'\varphi(L)), \tag{5.26b}$$

is equivalent to (5.25),

4. *the solution of* (5.25) *with the initial condition $L(0) = L_0$ can be written in two forms*

$$L(t) = \tau(G_+(t))L_0G_+^{-1}(t) = \tau(G_-(t))L_0G_-^{-1}(t), \tag{5.27}$$

where flows $F(t)$ and $G(t)$ are solutions of the following initial-value problems:

$$\dot{G}_+ = N_+(L)G_+, \quad G_+(0) = \mathbb{1}, \tag{5.28a}$$

$$\dot{G}_- = N_-(L)G_-, \quad G_-(0) = \mathbb{1}. \tag{5.28b}$$

Moreover, these flows are related to the solution $L(t)$ *of* (5.25) *with the initial condition* $L(0) = L_0$ *in the following way*:

$$G_-^{-1}(t)G_+(t) = \mathrm{e}^{tD\varphi(L_0)}, \tag{5.29a}$$

$$G_+(t)\,\mathrm{e}^{D\varphi(L_0)}G_+^{-1}(t) = G_-(t)\,\mathrm{e}^{D\varphi(L_0)}\,G_-^{-1}(t) = \mathrm{e}^{D\varphi(L(t))}. \tag{5.29b}$$

Proof. Proofs of particular parts of this theorem are analogous to those of the previous theorem. □

Finally we analyze Hamiltonian equations generated by a Hamilton function φ which is $\mathcal{G}$-invariant with respect to $\Phi^\kappa(G, L) = GL\kappa(G)$. We denote the set of these invariant functions by $\mathcal{I}_\kappa(\mathcal{G})$. Additionally, we suppose that anti-automorphism κ of $\mathfrak{g}$ is orthogonal with respect to the non-degenerate invariant scalar product in $\mathfrak{g}$ and commutes with the unitary classical R matrix. Then it is easy to obtain two relations

$$D'\varphi(L) = \kappa(D\varphi(L)), \tag{5.30a}$$

$$\forall\, L_0, G \in \mathcal{G}, \quad D\varphi(GL_0\kappa(G)) = \mathrm{Ad}_G(D\varphi(L_0)), \tag{5.30b}$$

valid for any $\mathcal{G}$-invariant function $\varphi \in \mathcal{I}_\kappa(\mathcal{G})$. These equalities are useful in the proof of the next theorem.

Theorem 15. *Let* $\varphi \in \mathcal{I}_\kappa(\mathcal{G})$ *and Lie algebra* $\mathfrak{g}$ *be equipped with a non-degenerate invariant scalar product. Additionally, suppose that orthogonal anti-automorphism* κ *of* $\mathfrak{g}$ *commutes with a unitary classical R matrix. Then*

1. *Hamiltonian equation* (5.8a) *has the form*

$$\dot{L} = N(L)L + L\kappa(N(L)), \quad N(L) = \tfrac{1}{2}R(D\varphi(L)), \tag{5.31}$$

2. *any two functions* $\varphi_1, \varphi_2 \in \mathcal{I}_\kappa(\mathcal{G})$ *are in involution with respect to the Sklyanin bracket* (5.6),
3. *each equation*

$$\dot{L} = N_+(L)L + L\kappa(N_+(L)), \quad N_+(L) = R_+(D\varphi(L)), \tag{5.32a}$$

$$\dot{L} = N_-(L)L + L\kappa(N_-(L)), \quad N_-(L) = R_-(D\varphi(L)), \tag{5.32b}$$

   is equivalent to (5.31),
4. *the solution of* (5.31) *with the initial condition* $L(0) = L_0$ *can be written in two forms*

$$L(t) = G_+(t)L_0\kappa(G_+(t)) = G_-(t)L_0\kappa(G_-(t)), \tag{5.33}$$

   where flows $F(t)$ *and* $G(t)$ *are solutions of the following initial-value problems*

$$\dot{G}_+ = N_+(L)G_+, \quad G_+(0) = \mathbb{1}, \tag{5.34a}$$

$$\dot{G}_- = N_-(L)G_-, \quad G_-(0) = \mathbb{1}. \tag{5.34b}$$

Moreover, these flows are related to the solution $L(t)$ *of* (5.31) *with the initial condition* $L(0) = L_0$ *in the following way*:

$$G_-^{-1}(t)G_+(t) = \mathrm{e}^{tD\varphi(L_0)}, \tag{5.35a}$$

$$G_+(t)\,\mathrm{e}^{D\varphi(L_0)}\,G_+^{-1}(t) = G_-(t)\,\mathrm{e}^{D\varphi(L_0)}\,G_-^{-1}(t) = \mathrm{e}^{D\varphi(L(t))}. \tag{5.35b}$$

Proof. The proof is analogous to those of two previous theorems. □

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